Superconvergence: An Old Field with New Territories

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Acknowledgement: NSF grants DMS-1115530, DMS-1419040
NSFC grants 11471031, 91430216; and NASF U1530401

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Irish Numerical Analysis Forum (INAF), May 20, 2021

- What is superconvergence, anyway?
- Why superconvergence?

1) Cost Reduction! Tremendous gain under favorable situation!

2) A posteriori error estimates

recovery type versus residual type error estimators

industrial application versus academic development

3) Other benefits, such as enhancing eigenvalue approximation, ...

4) Industrial Application:

Superconvergence Patch Recovery (SPR) by Zienkiewicz-Zhu 1992

Polynomial Preserving Recovery (PPR) by Naga-Zhang 2005

- SPR: FE software packages ANSYS, LS-DYNA, Abaqus, Diffpack,......
- PPR has been adopted by COMSOL Multiphysics (FEMLAB) since 2008

"To get better accuracy, select the Recover check box. COMSOL Multiphysics then evaluates the derivatives (and u itself) using a polynomial-preserving recovery technique by Z. Zhang (see Ref. 1)."

– COMSOL MultiPhysics 3.5a User's Guide p.471, 2008

"In the Expression text field enter the function ppr(solid.mises). The function ppr() corresponds to the Recover setting in the earlier note on page 42 for Surface plots. The Recover setting with the ppr function is used to increase the quality of the stress field results. It uses a polynomial-preserving recovery (ppr) algorithm, which is a higher-order interpolation of the solution on a patch of mesh elements around each mesh vertex."

– Introduction to COMSOL Multiphysics Version 4.4, November 2013, p.45;

Versions 5.0—5.6, 2014–2020.

- Where are we standing?

What do we know of superconvergence?

A lot for traditional h-version FEMs from past 50 years: **An old field**

What do we not know of superconvergence?

Very little (comparing with the h-version FEM) **New territories**!

p-version finite element methods

finite volume methods

spectral and spectral collocation methods

discontinuous Galerkin methods

Nonconforming finite element methods

....

Polynomial Spectral Collocation – for ODEs (Z. Zhang, SINUM 2012)

Fundamentals: Natural supercovergence points and approximation theory

Approach: Analysis based on orthogonal polynomials

We interpolate (or collocate) u at a set of N special points on $[-1,1]$:

$$u_N(x_k) = u(x_k), \quad -1 \le x_1 < \cdots < x_N \le 1.$$

Goal: Identify y_j's, where u_N' superconverges to u' in the sense that

$$N^\alpha |(u-u_N)'(y_j)| \le C \max_{x\in[-1,1]} |(u-u_N)'(x)|, \quad \alpha > 0.$$

Definition of suprconvergence points for the p-version and spectral methods:

y_j's are independent of the particular choice of u: derivative superconvergence points (of the interpolation) for a class of functions.

A set of popular interpolation points are roots of:

2.1) L_N, Legendre polynomial of degree N: Gauss points

2.2) $L_N - L_{N-2}$, Lobatto polynomial: Gauss-Lobatto points

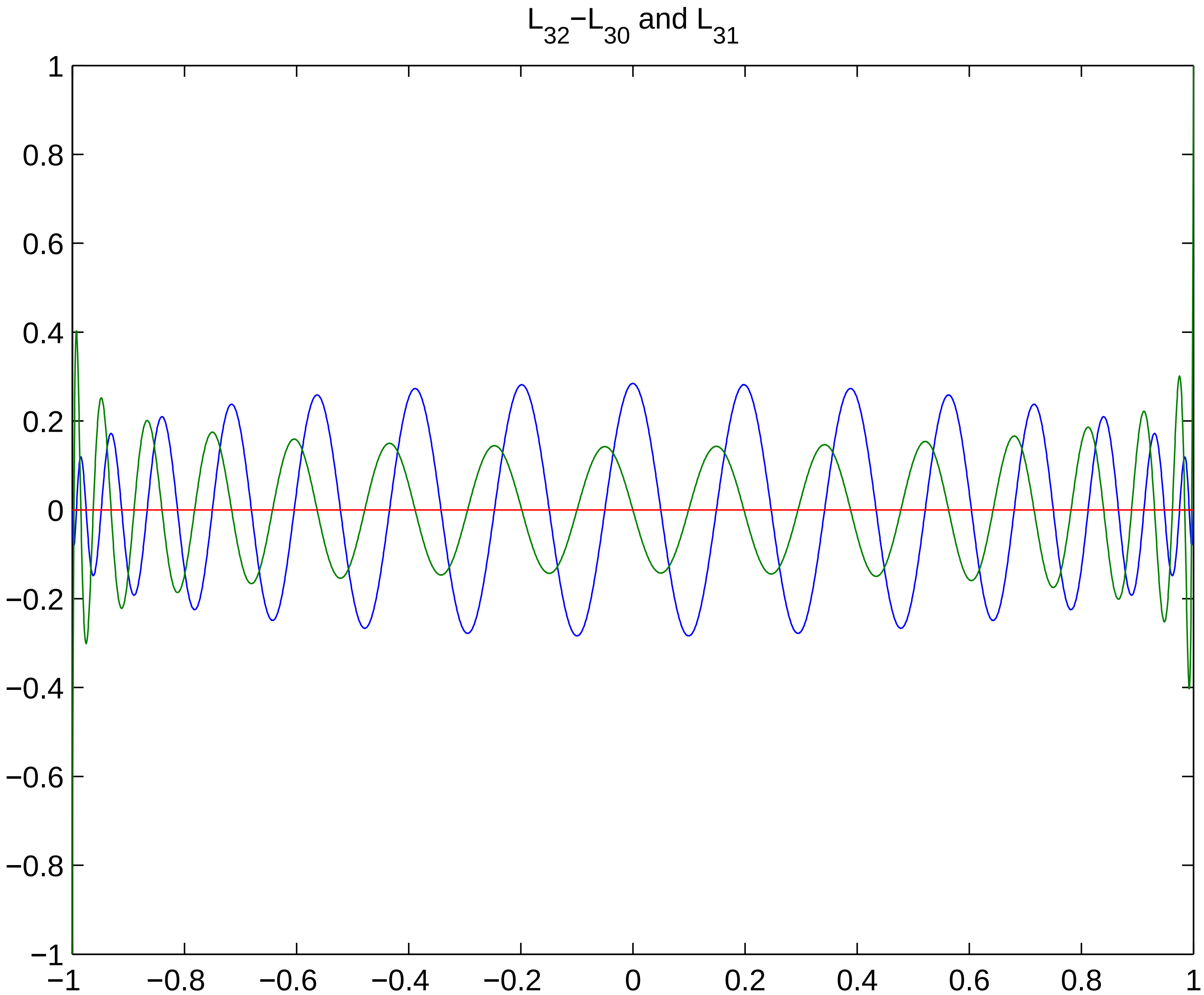
2.3) $L_N + L_{N-1}$, left Radau polynomial: left Radau points

2.4) $L_N - L_{N-1}$, right Radau polynomial: right Radau points

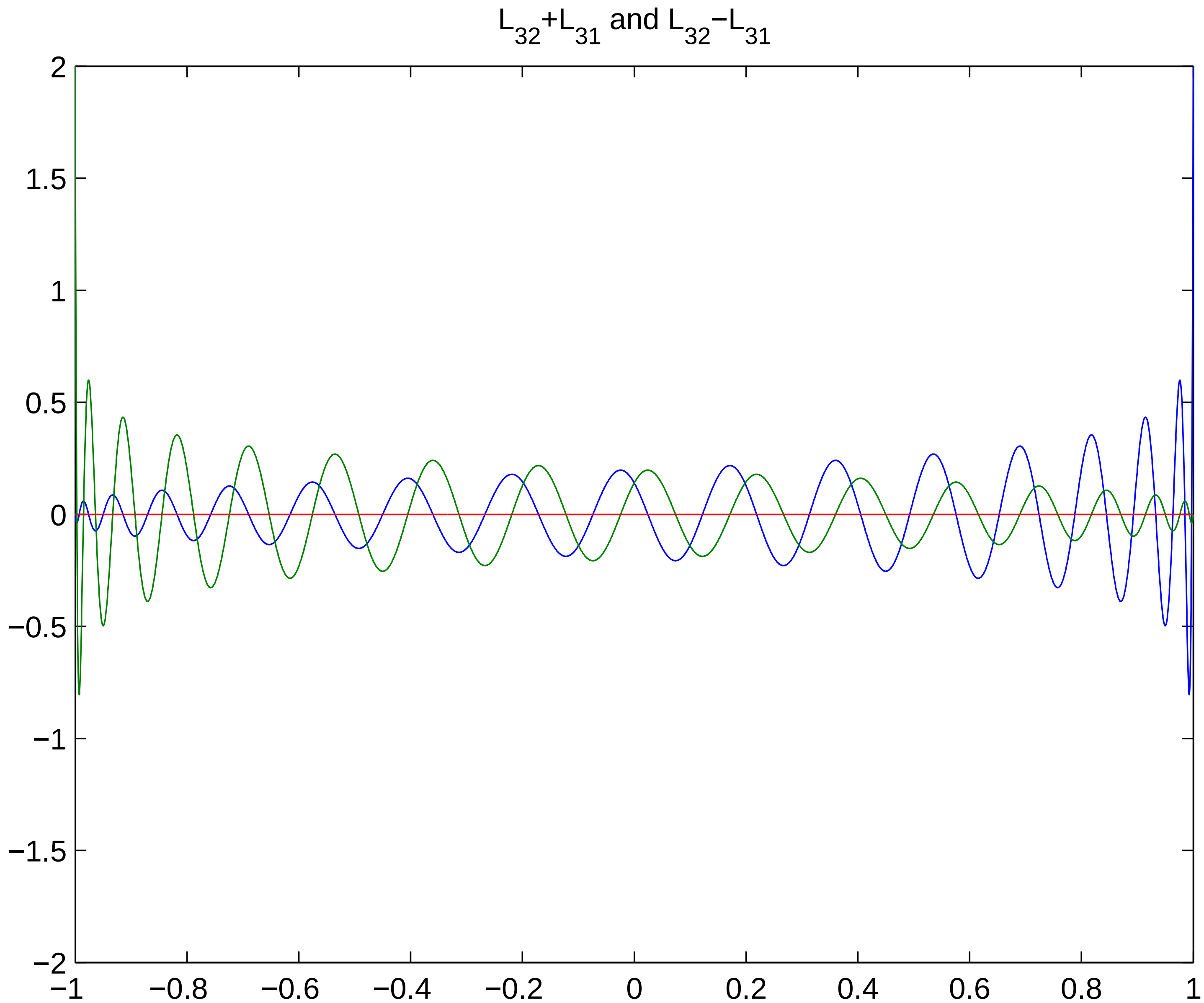
Some important relations (similar for other Jacobi polynomials):

$$(2N-1)L_{N-1}(x) = (L_N - L_{N-2})'(x),$$

$$\frac{1}{2N-1}(L_N - L_{N-2})(x) = \frac{1}{N(N-1)}(x^2-1)L'_{N-1}(x);$$
$$N(L_N + L_{N-1})(x) = (x+1)(L_N - L_{N-1})'(x),$$
$$N(L_N - L_{N-1})(x) = (x-1)(L_N + L_{N-1})'(x).$$



$L_{32}-L_{30}$ and L_{31}
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


L32+L31 and L32−L31
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

These relations help us to locate derivative zero points

- Derivative zeros of the Legendre polynomial of degree N is the interior zeros of the Lobatto polynomial of degree $N+1$

- Derivative zeros of the Lobatto polynomial of degree N is the zeros of the Legendre polynomial of degree $N-1$

- Derivative zeros of the right (left) Radau polynomial are the left (right) Radau points (except ± 1) of the same degree

- Superconvergence points are problem and method dependent

- A properly designed method uses correct basis functions, then the problem is narrow down to interpolation

Let x_ks be roots of particular orthogonal polynomials and

$$u_N'(x_k) = u'(x_k) = f(x_k), \quad u_N(-1) = u(-1).$$

• Interpolating derivative at the N Gauss points, the function value approximation is superconvergent at the $N-1$ interior Lobatto points.

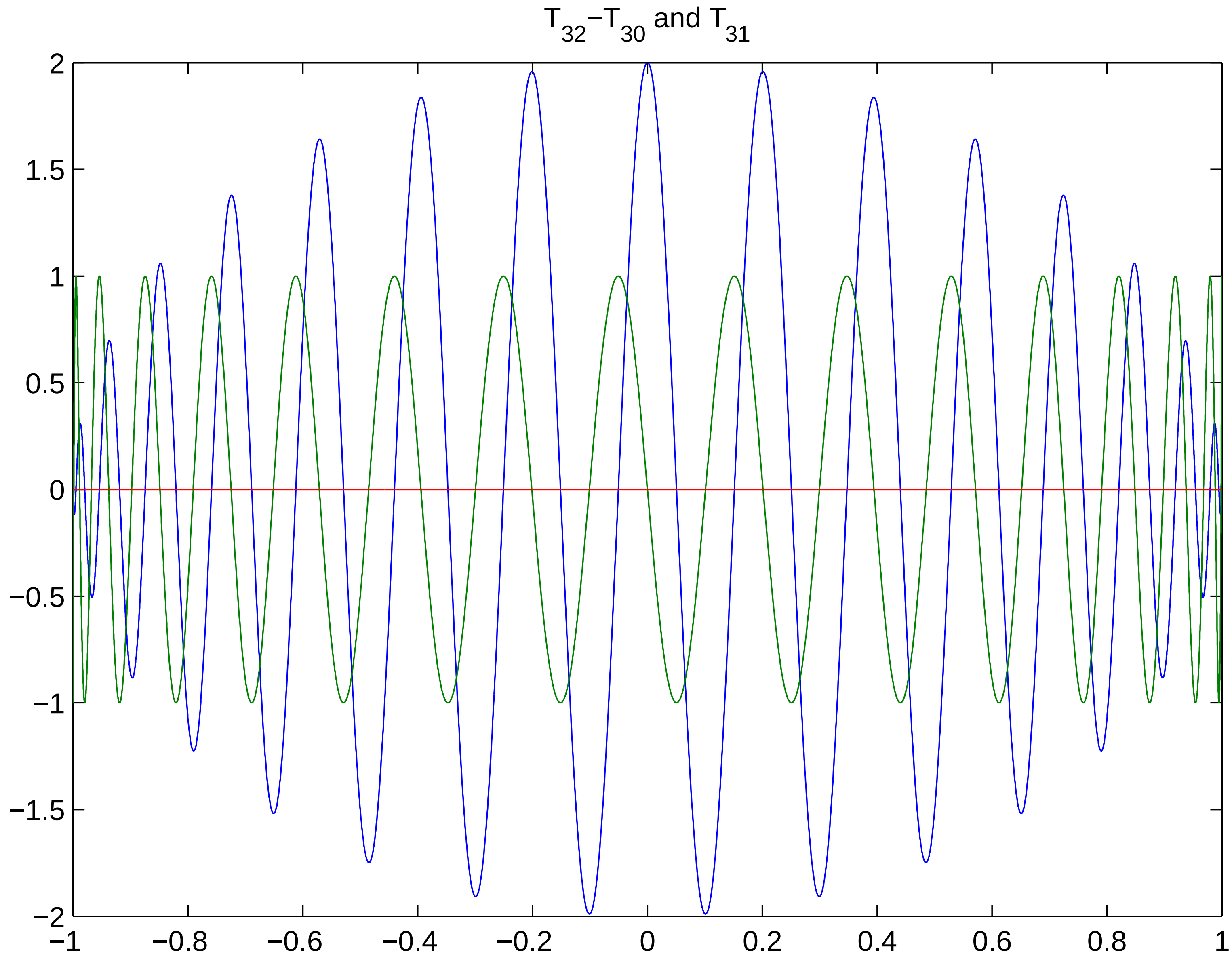
• Interpolating derivative at the N Lobatto points, function value approximation is superconvergent at the $N-1$ Gauss points.

• Interpolating derivative at the N left (right) Radau points, the function value approximation is superconvergent at the $N-1$ right (left) Radau points.

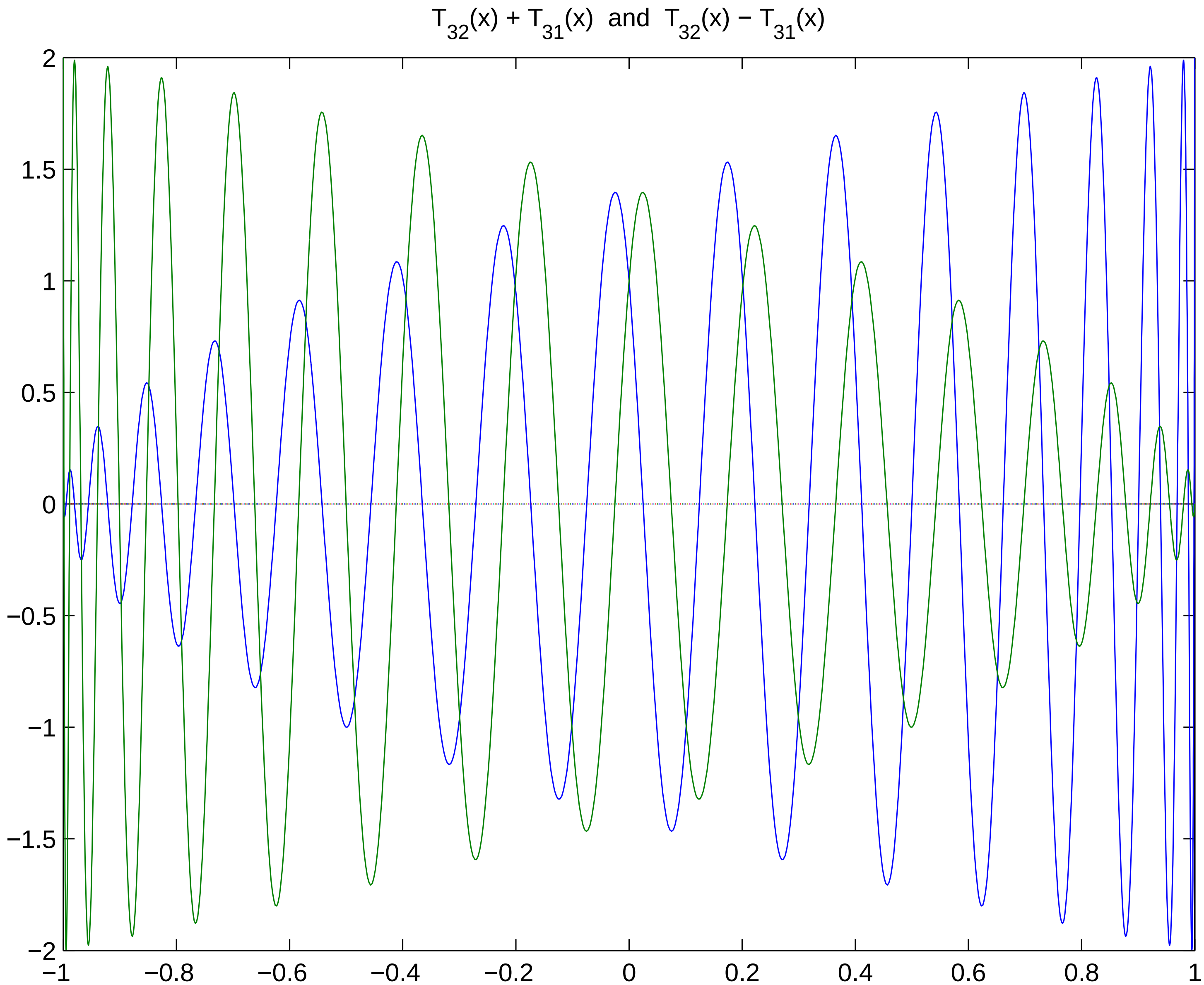
Somilar results for Jacobi polynomials and generalized Jacobi functions

Li-Lian Wang, Xiaodan Zhao, and Zhimin Zhang, JSC 2014

Xuan Zhao and Zhimin Zhang, SISC 2017



$T_{32}-T_{30}$ and T_{31}
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1


T32(x) + T31(x) and T32(x) − T31(x)
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1

Numerical Test Examples

Examples 1 (Runge' example).

$$u'(x) = f(x) = \frac{1}{1+25x^2}, \qquad \rho = \frac{\sqrt{5^2+1}+1}{5} \approx 1.2198.$$

It has two single poles at $\pm i/5$.

Example 2.

$$u'(x) = f(x) = \frac{1}{2-x}, \qquad \rho = 2+\sqrt{3} \approx 3.7321.$$

It has a single pole at $x = 2$.

We expect faster convergence for Example 2 compared with Example 1.

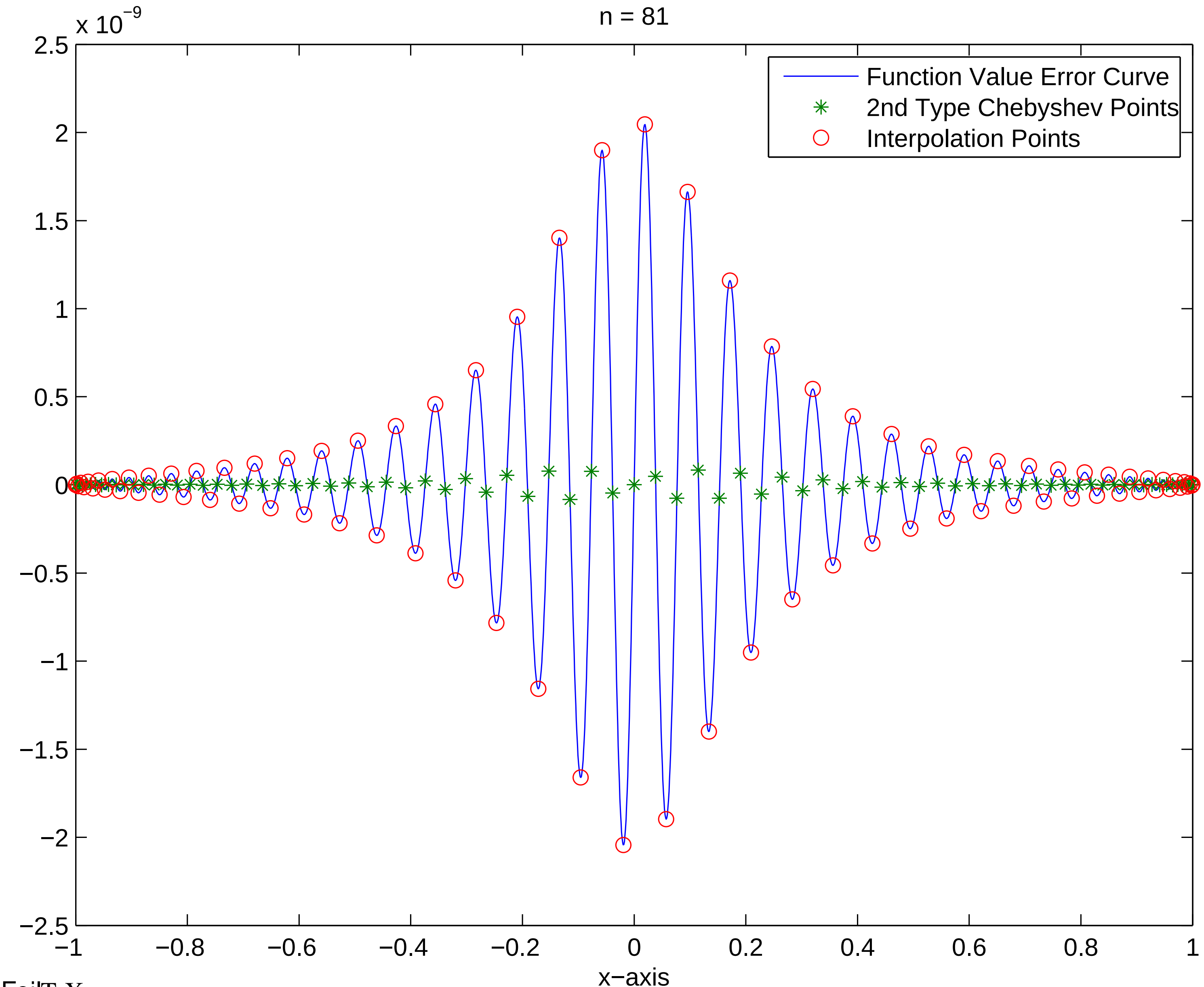


Figure 1: Derivative interpolation at the Chebyshev points, Example 1

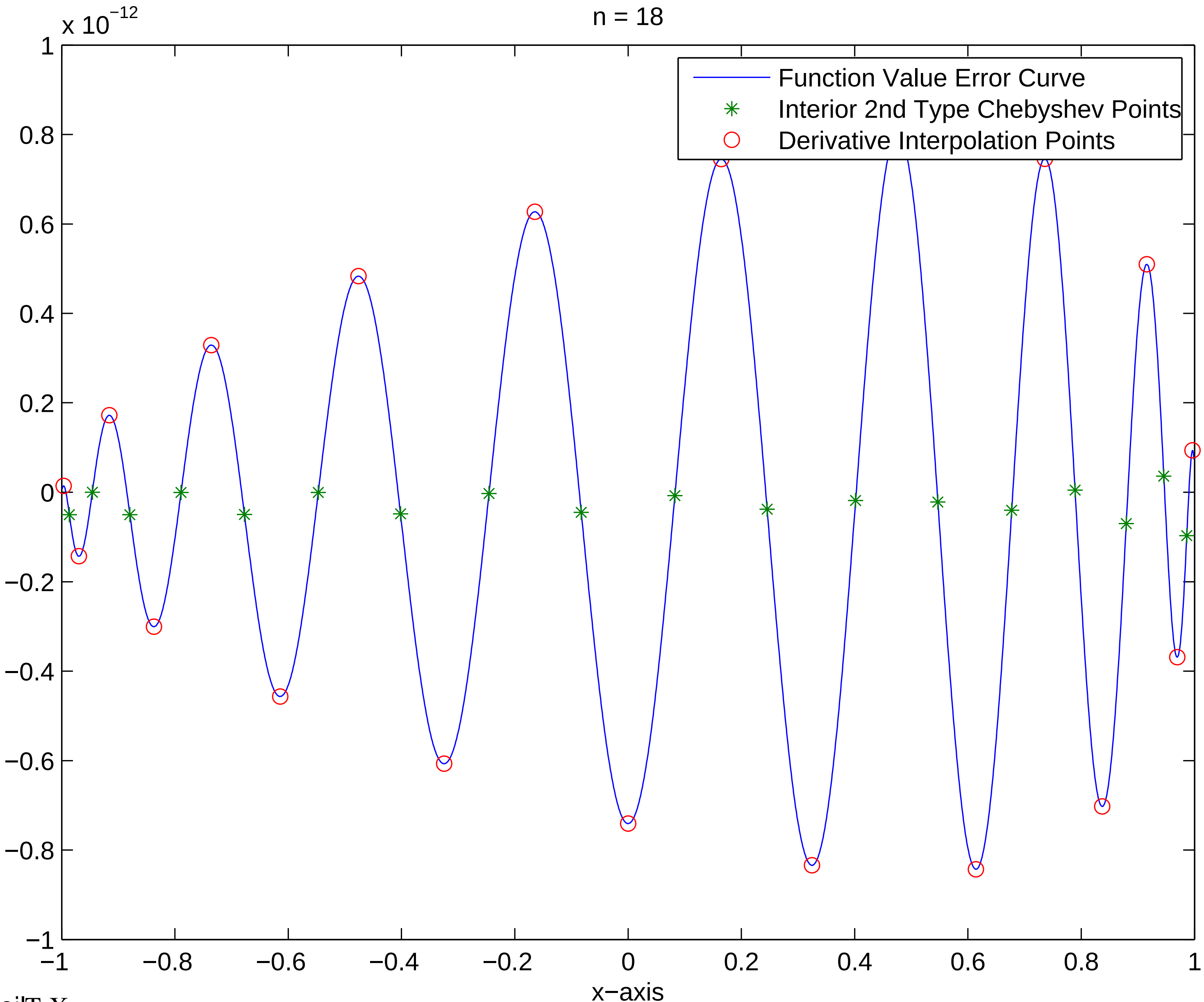


Figure 2: Derivative interpolation at the Chebyshev points, Example 2

LDG for 1D linear conservation laws

• An earlier discovery from numerical tests and $2k+1$ order in negative norm and post-processed solution in L_2-norm for $u_t + u_x = 0$:

Bernardo Cockburn, Mitchell Luskin, Chi-Wang Shu, Endre Süli, Enhanced accuracy by post-processing for finite element methods for hyperbolic equations, Mathematics of Computation 72 (2003), 577-606.

• Recent theoretical development: semi-discretization

Yang Yang and Chi-Wang Shu proved point-wise $k+2$ rate.

Analysis of optimal supercovergence of discontinuous Galerkin method for linear hyperbolic equations, SIAM J. Numer. Anal., 50: 3110-3133, 2012

• **Point-wise** $2k+1$ **rate has remained open till 2014**!!

Boundary condition: $u(0,t) = u(2\pi, t)$ or $u(0,t) = g(t)$

Correction idea

- Goal: Design a correction function w to improve the error $a(u - P_h^- u, v)$

  such that for some $l \geq 1$, $\left|a(u - P_h^- u + w, v)\right| \lesssim h^{k+1+l}$, $\forall v \in V_h$
- Ultimate goal: $l = k$ and $\|u_h - u_I\|_0 \lesssim h^{2k+1}$ with $u_I = P_h^- u - w$
- Difference of the new design from that of the FEM:

  Explicit vs Implicit: Constructive proof and existence proof

  "practical FEM" and "theoretical FEM"

  Semi-discretization and time dependent feature

  Explicit construction: of w^l for any $1 \leq l \leq k$ such that $w^l(x^-_{j+\frac{1}{2}}) = 0$ and $\|w^l\|_{0,\infty} \lesssim h^{k+2}\|u\|_{k+4,\infty}$. Moreover, if $u \in W^{k+l+2,\infty}, 1 \leq l \leq k$,

$$\left|a(u - P_h^- u + w^l, v)\right| \lesssim h^{k+1+l}\|u\|_{k+l+2,\infty}\|v\|_{0,1}$$

If the initial value is chosen such that

$$\|u_h(\cdot,0) - u_I^l(\cdot,0)\|_0 \lesssim h^{k+l+1}\|u\|_{k+l+2,\infty}, \quad u_I^l = P_h^- u + w^l,$$

we have $\|u_h - u_I^l\|_0 \lesssim (1+t)h^{k+l+1}\|u\|_{k+l+2,\infty}$.

- $\|u_h - P_h^- u\|_0 \lesssim (1+t)h^{k+2}\|u\|_{k+4,\infty}$
- $\|u_h - u_I^k\|_0 \lesssim (1+t)h^{2k+1}\|u\|_{2k+2,\infty}$. super-closeness
- Superconvergence rate is very sensitive to the method of initialization

Waixiang Cao, Zhimin Zhang, and Qingsong Zou, Superconvergence of Discontinuous Galerkin method for linear hyperbolic equations *SIAM Journal on Numerical Analysis* 52-5 (2014), 2555-2573.

Waixiang Cao and Zhimin Zhang, Point-wise and cell average error estimates of the DG and LDG methods for 1D hyperbolic and parabolic equations (in Chinese), *Sci Sin Math*, 45 (2015) 1115-1132.

Superconvergence (Sharp!)

- Downwind points (point-wise): $(u - u_h)(x^-_{j+\frac{1}{2}}, t) = O(h^{2k+1})$
- Cell average (discrete L^2 norm):

$$\|e_u\|_c = \left(\frac{1}{N} \sum_{j=1}^{N} \Big(\frac{1}{h_j} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (u - u_h) \Big)^2 \right)^{\frac{1}{2}} = O(h^{2k+1})$$

- Derivative value at interior left Radau points:

$$\partial_x (u - u_h)(R^l_{j,l}, t) = O(h^{k+1})$$

- Function value at right Radau points:

$$(u - u_h)(R^r_{j,l}, t) = O(h^{k+2})$$

Numerical test

Example 1: $u_t + u_x = 0,\ u_0(x) = e^{\sin(x)},\ u(0,t) = u(2\pi, t)$

Piecewise uniform mesh composed by dividing $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, 2\pi]$

into $N/2$ subintervals with $N = 4, \ldots, 512$

e_1: maximal error at downwind points

e_2: discrete L^2 error at downwind points

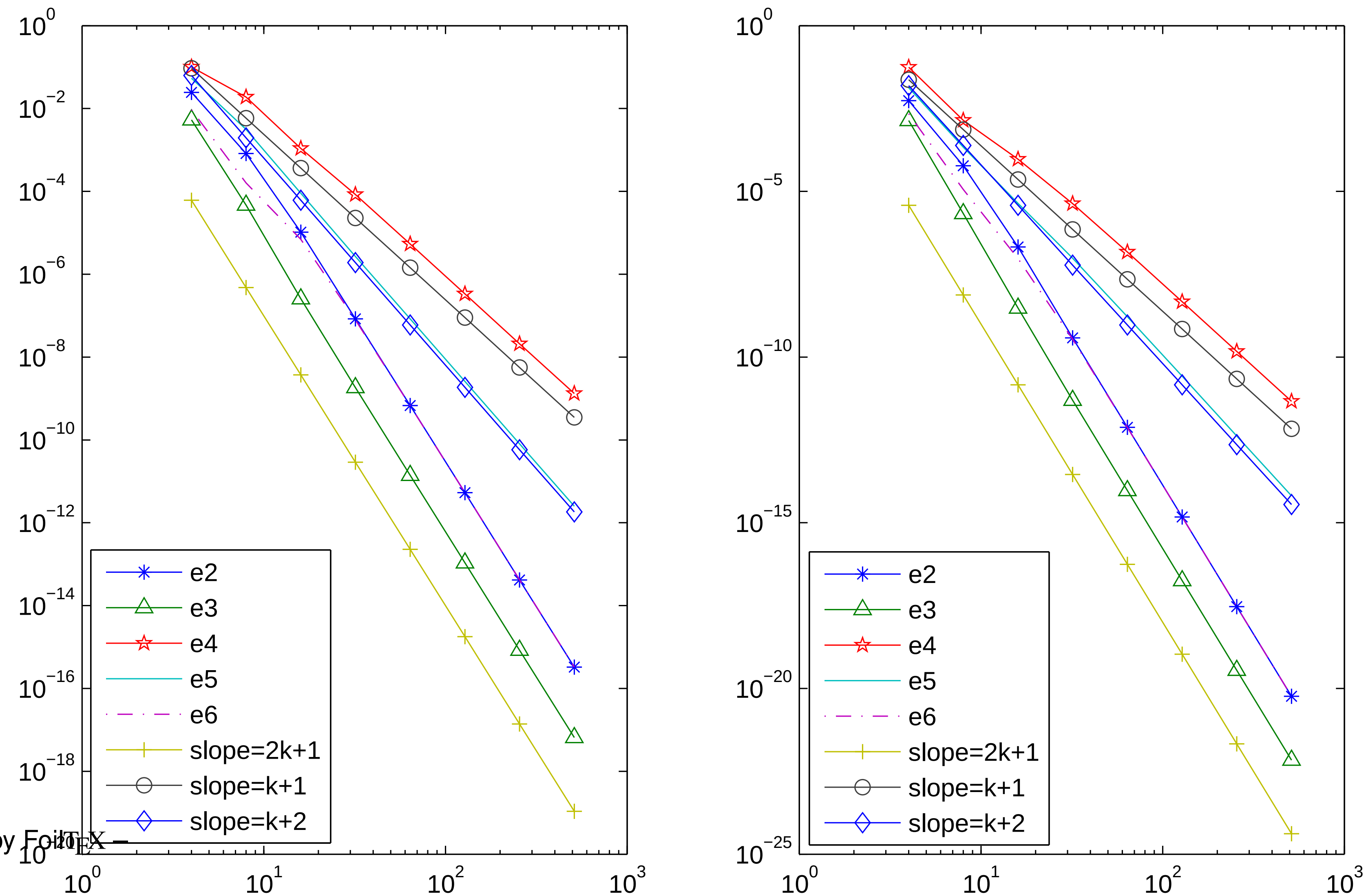
e_3: domain average

e_4: maximal derivative error at interior left Radau points

e_5: maximal function value error at right Radau points

e_6: cell average

Figure 3: left: $k = 3$, right: $k = 4$



Comparison of four different initial discretizations

Method 1 : $u_h(x,0) = R_h u(x,0)$ (L^2 projection of u)

Method 2 : $u_h(x,0) = (P_h^- u)(x,0)$

Method 3 : $u_{ht}(x,0) = (P_h^- u_t)(x,0),\ u_h(x^-_{j+\frac{1}{2}},0) = (P_h^- u)(x^-_{j+\frac{1}{2}},0)$

Method 4 : $u_h(x,0) = u_I^k(x,0)$

Table 1: $k = 4$

N	Method 1		Method 2		Method 3		Method 4	
	e_1	rate	e_1	rate	e_1	rate	e_1	rate
2	1.43e-04	–	8.26e-05	–	1.09e-04	–	5.25e-05	–
4	2.69e-05	2.4	2.02e-06	5.4	1.00e-06	6.8	3.66e-07	7.2
8	7.85e-07	5.1	1.25e-08	7.3	1.10e-08	6.5	7.60e-10	8.9
16	2.02e-08	5.3	9.26e-11	7.1	7.71e-11	7.1	1.51e-12	9.0
32	3.81e-10	5.7	5.13e-12	4.2	3.72e-13	7.7	2.96e-15	9.0
64	4.78e-12	6.3	7.23e-14	6.2	1.52e-15	7.9	5.80e-18	9.0

LDG method for 1D parabolic equations

$$u_t = u_{xx}, \quad (x,t) \in [0,2\pi] \times (0,T], \quad u(x,0) = u_0(x) \tag{1}$$

Boundary conditions: 1) $u(0,t) = u(2\pi,t)$, or

2) $u(0,t) = g_0(t), u_x(2\pi,t) = g_1(t)$, or

3) $u_x(0,t) = g_0(t), u(2\pi,t) = g_1(t)$

Equation (1) as a first order linear system: $u_t = q_x, \quad q = u_x$

LDG Methods: find $(u_h, q_h) \in V_h$ such that

$$(u_{ht}, v)_j = -(q_h, v_x)_j + \hat{q}_h v^-|_{j+\frac{1}{2}} - \hat{q}_h v^+|_{j-\frac{1}{2}}$$
$$(q_h, w)_j = -(u_h, w_x)_j + \hat{u}_h w^-|_{j+\frac{1}{2}} - \hat{u}_h w^+|_{j-\frac{1}{2}}$$

Numerical fluxes $\hat{u}_h, \hat{q}_h$ are taken as alternating fluxes

$$\hat{u}_h = u_h^-, \quad \hat{q}_h = q_h^+ \tag{2}$$

or

$$\hat{u}_h = u_h^+, \quad \hat{q}_h = q_h^- \tag{3}$$

Note: We test

a) both (2) and (3) in the periodic boundary condition 1)

b) (2) in the mixed boundary condition 2)

c) (3) in the mixed boundary condition 3)

Correction function for parabolic equations

• Different from FEM and FVM for elliptic equations: semi-discretization and time dependent feature.

• Different from DG methods for hyperbolic equations: correction functions for both variables u and q have to be constructed simultaneously.

• Main difficulty: Balance the interplay between two correction functions.

Waixiang Cao and Zhimin Zhang, Superconvergence of Local Discontinuous Galerkin method for one-dimensional linear parabolic equations, *Mathematics of Computation* 85 (2016), 63-84.

Waixiang Cao and Zhimin Zhang, Point-wise and cell average error estimates of the DG and LDG methods for 1D hyperbolic and parabolic equations (in Chinese), *Sci Sin Math*, 45 (2015) 1115-1132.

Superconvergence

- Downwind and upwind points (point-wise):

$$(u-\hat{u}_h)(x_{j-\frac{1}{2}},t)=O(h^{2k+1}),\quad (q-\hat{q}_h)(x_{j-\frac{1}{2}},t)=O(h^{2k+1})$$

- Cell average: $\|e_u\|_c=O(h^{2k+1}),\quad \|e_q\|_c=O(h^{2k+1})$
- Domain average:

$$\frac{1}{2\pi}\int_0^{2\pi}(u-u_h)dx=O(h^{2k+1}),\quad \frac{1}{2\pi}\int_0^{2\pi}(q-q_h)dx=0\qquad \text{periodic}$$

$$\frac{1}{2\pi}\int_0^{2\pi}(u-u_h)dx=O(h^{2k+1}),\quad \frac{1}{2\pi}\int_0^{2\pi}(q-q_h)dx=O(h^{2k+1})\qquad \text{mixed}$$

• Function value approximation at Radau points:

$$(u-u_h)(R^r_{j,N},t)=O(h^{k+2}),\ \ (q-q_h)(R^l_{j,N},t)=O(h^{k+2})\quad \text{fluxes (2)}$$

$$(u-u_h)(R^l_{j,N},t)=O(h^{k+2}),\ \ (q-q_h)(R^r_{j,N},t)=O(h^{k+2})\quad \text{fluxes (3)}$$

• Derivative value approximation at interior Radau points:

$$\partial_x(u-u_h)(R^l_{j,N},t)=O(h^{k+1}),\ \ \partial_x(q-q_h)(R^r_{j,N},t)=O(h^{k+1})\quad \text{fluxes (2)}$$

$$\partial_x(u-u_h)(R^r_{j,N},t)=O(h^{k+1}),\ \ \partial_x(q-q_h)(R^l_{j,N},t)=O(h^{k+1})\quad \text{fluxes (3)}$$

Note: Numerical observation: $O(h^{k+2})$

• In a purely numerical method, mathematics demonstrates her beauty and elegant in symmetry and harmony!

Numerical test

Example 2: $u_t = u_{xx}$ with initial condition $u_0(x) = \cos(x) + e^{x+1}$ and

boundary condition: $u_x(0,0) = e^{t+1}, \quad u(2\pi,0) = e^{-t} + e^{2\pi+t+1}$.

Numerical fluxes: $\hat{u}_h = u_h^+,\ \hat{q}_h = q_h^-$

Uniform mesh with $N = 2^m$ ($m = 2, 3, \ldots, 6$) for $k = 3$

Errors:

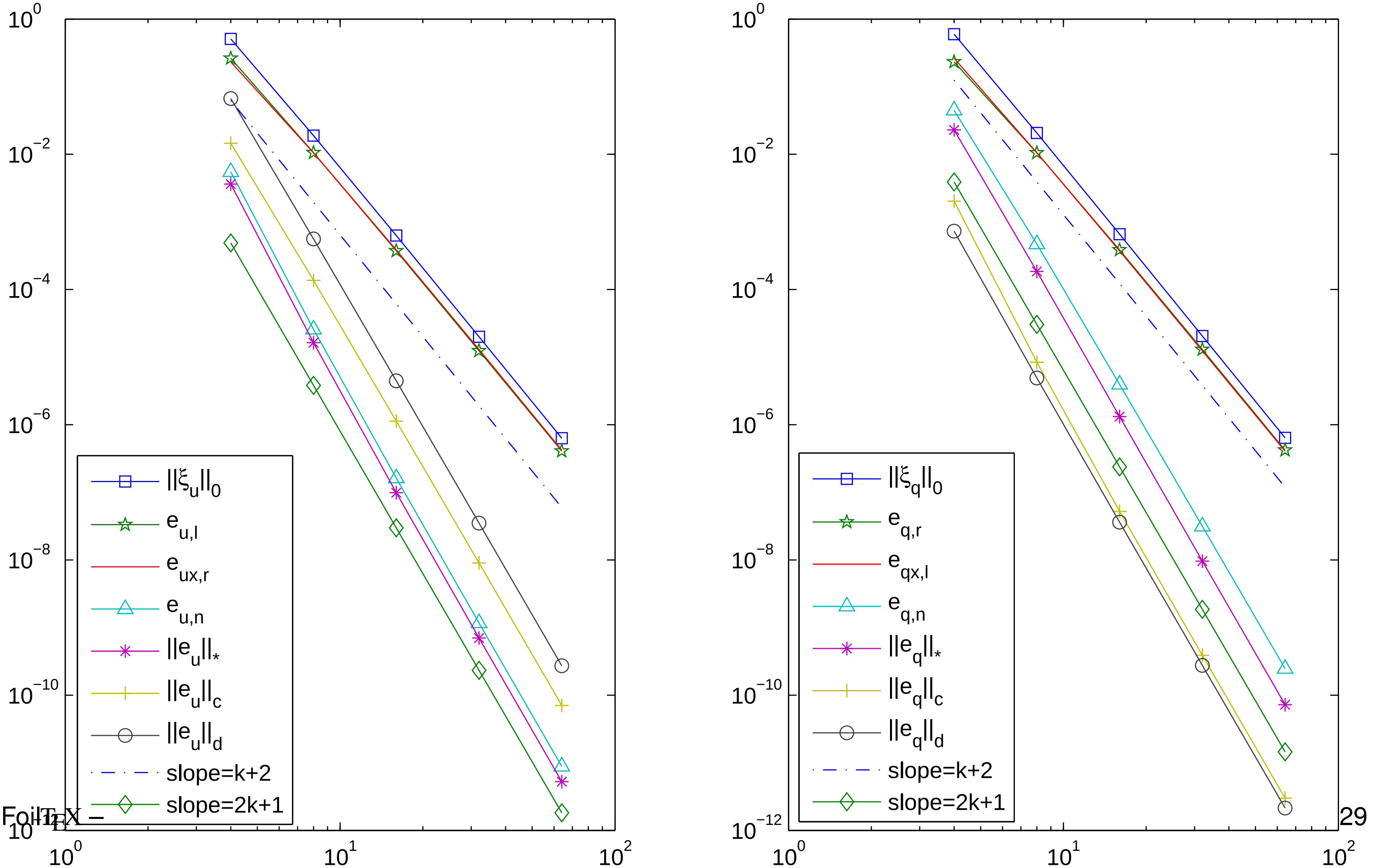
$$\|\xi_u\|_0 = \|u - P_h^+ u\|_0, \quad \|\xi_q\|_0 = \|q - P_h^- q\|_0$$

$$e_{u,n} = \max|(u-u_h)(x^+_{j-\frac{1}{2}}, t)|, \quad e_{q,n} = \max|(q-q_h)(x^-_{j-\frac{1}{2}}, t)|$$

$$e_{u,l} = \max|(u-u_h)(R^l_{j,N}, t)|, \quad e_{q,r} = \max|(q-q_h)(R^r_{j,N}, t)|$$

$$e_{ux,r} = \max|\partial_x(u-u_h)(R^r_{j,N}, t)|, \quad e_{qx,l} = \max|\partial_x(q-q_h)(R^l_{j,N}, t)|$$

Figure 4: Error curves in the mixed boundary condition for $k = 3$



2D case

Model problem: $u_t + u_x + u_y = 0$, $u(x,y,0) = u_0(x,y)$ with boundary condition.

DG method (tensor-product space, rectangular elements):

$$a_\tau(u_h, v) = \int_\tau (u_{ht}v - u_h(v_x + v_y)) + \int_{\partial\tau} \hat{u}_h v = 0, \quad \hat{u}_h = u_h^-.$$

- Periodic boundary condition:

$$\hat{u}_h(x_{\frac{1}{2}}, y) = u_h(x_{\frac{1}{2}}^-, y) = u_h(x_{m+\frac{1}{2}}^-, y) = \hat{u}_h(x_{m+\frac{1}{2}}, y)$$

$$\hat{u}_h(x, y_{\frac{1}{2}}) = u_h(x, y_{\frac{1}{2}}^-) = u_h(x, y_{n+\frac{1}{2}}^-) = \hat{u}_h(x, y_{n+\frac{1}{2}})$$

- Dirichlet boundary condition (correction discretization):

$$u_h(x_{\frac{1}{2}}^-, y) = u_I(x_{\frac{1}{2}}^-, y), \quad u_h(x, y_{\frac{1}{2}}^-) = u_I(x, y_{\frac{1}{2}}^-)$$

Superconvergence results

Nodes: $e_{u,d} = \left(\frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} (u - u_h)^2 (x^-_{i+\frac{1}{2}}, y^-_{j+\frac{1}{2}}, t) \right)^{\frac{1}{2}} = O(h^{2k+1})$

Cell average: $e_{u,c} = \left(\frac{1}{nm} \sum_{\tau \in \mathcal{T}_h} \left(\frac{1}{|\tau|} \int_\tau (u - u_h) \right)^2 \right)^{\frac{1}{2}} = O(h^{2k+1})$

Partial derivative approximation along the left Radau edges:

$$e_{u,l} = \max_{P \in \mathcal{E}^l_x} |\partial_x (u - u_h)(P,t)| + \max_{Q \in \mathcal{E}^l_y} |\partial_y (u - u_h)(Q,t)| = O(h^{k+1})$$

Function value approximation at the right Radau points:

$$e_{u,r} = \max_{P \in \mathcal{R}^r} |(u - u_h)(P,t)| = O(h^{k+2})$$

Numerical test:

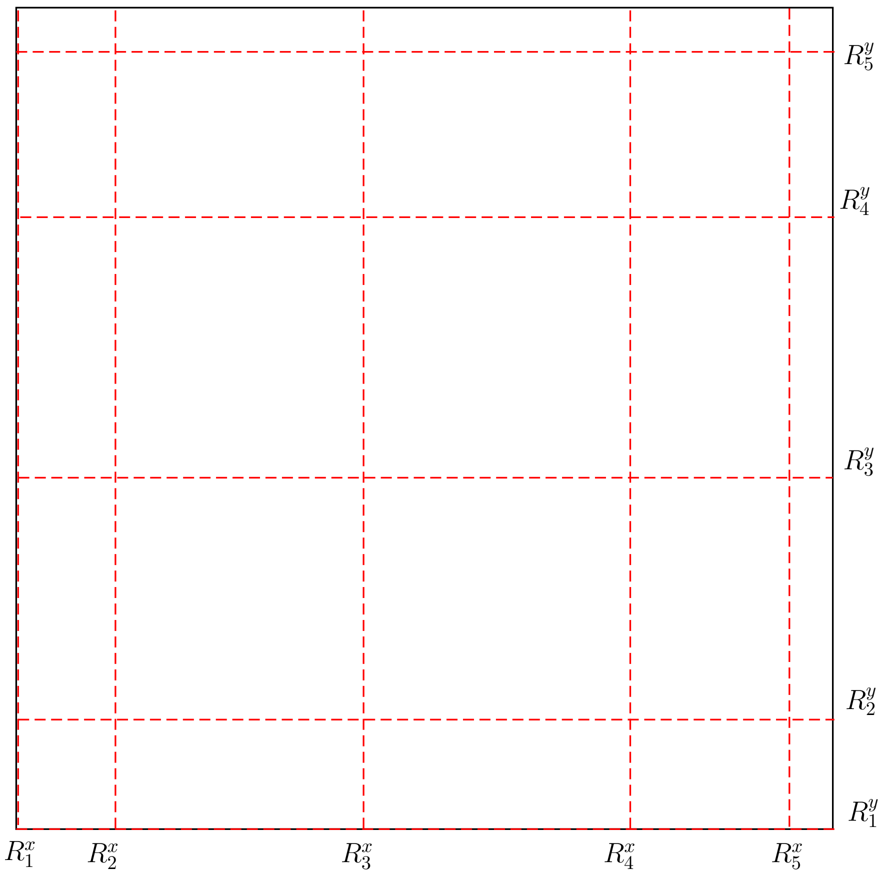
Example: $u_t + u_x + u_y = 0$, $u_0 = \sin(x+y)$

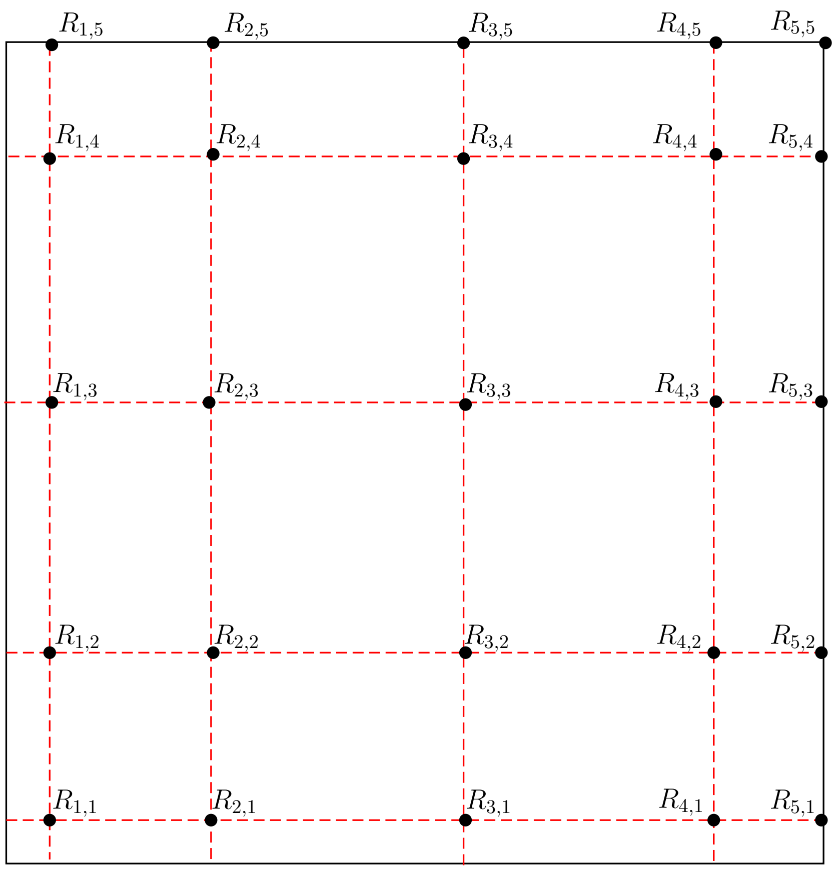
Mesh: Non-uniform meshes of $m \times n$ rectangles by randomly perturbing each node in the x and y axes of a uniform mesh by up to 20%

Note: Optimal convergence rates are observed and superconvergence phenomenon disappears using the L_2-projection to treat the initial condition.

Note: The special choice of the boundary discretization for DBC is to guarantee the superconvergence of the DG approximation. This is very different from the traditional L^2 projection or the Gauss-Radau interpolant.

Waixiang Cao, Chi-Wang Shu, Yang Yang, and Zhimin Zhang, Superconvergence of discontinuous Galerkin methods for 2D hyperbolic equations, *SIAM Journal on Numerical Analysis* 53-4 (2015), 1651-1671.


R_5^y
R_4^y
R_3^y
R_2^y
R_1^y
R_1^x
R_2^x
R_3^x
R_4^x
R_5^x


$R_{1,5}$
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$R_{5,2}$
$R_{1,1}$
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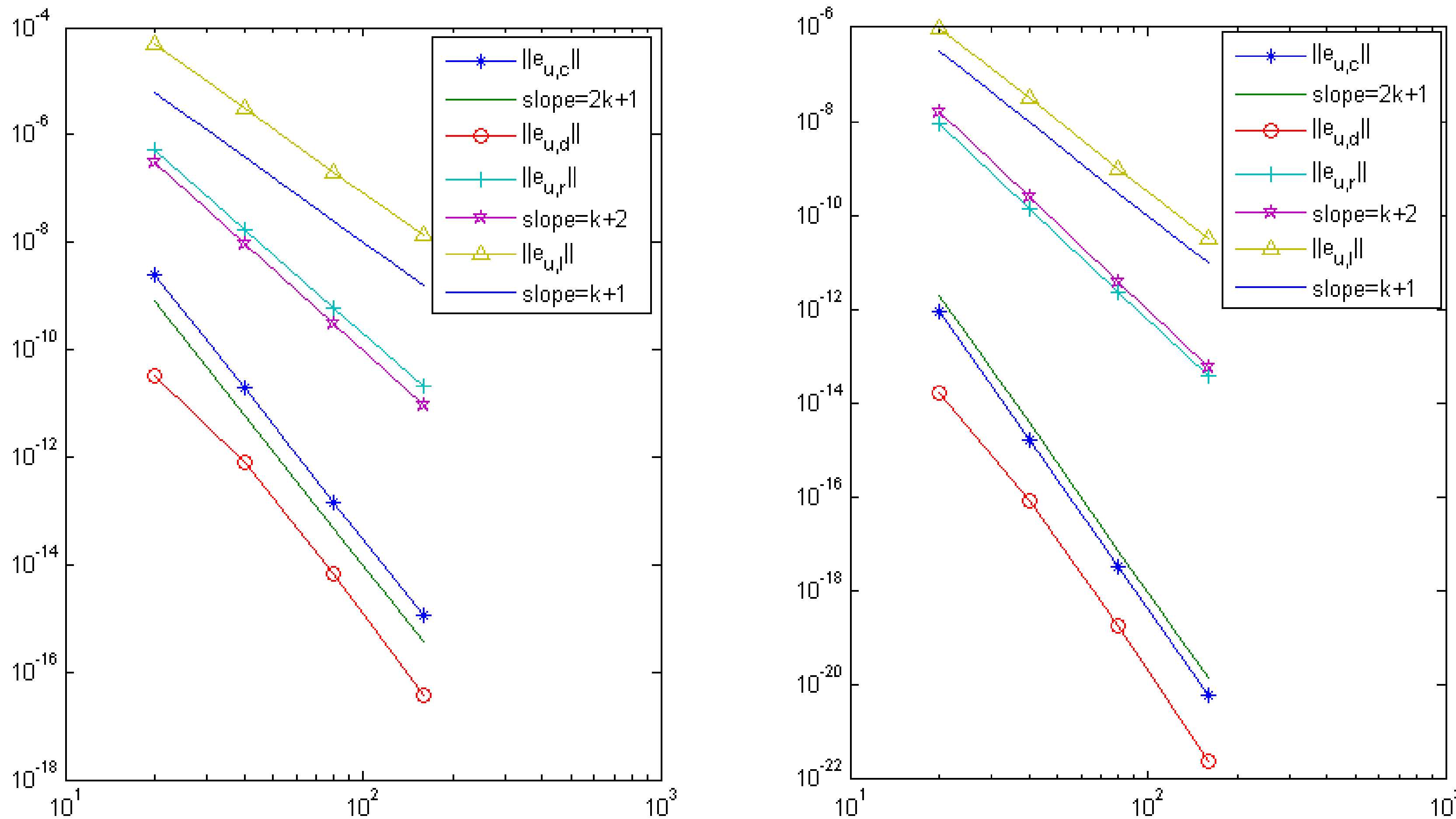
$R_{4,1}$
$R_{5,1}$

Figure 5: Errors with periodic boundary condition, left: $k = 3$, right: $k = 4$



Further Extension:

- DDG for $u_t + au_x = u_{xx}$ (semi-discretization): Nodal h^{2k} etc.

  Direct DG method without introducing $q = u_x$

  W. Cao, H. Liu, and Z.Z., Numer. Meth. PDEs 2017

- LDG for $u_{tt} = u_{xx} + au$ (semi-discretization): h^{2k+1} etc.

  W. Cao, D. Li, and Z.Z., CiCP 2017

- Immersed FEM for $-(au')' + bu' + cu = f$ with interface: Nodal h^{2k} etc.

Using weight function a^{-1} to define generalized Legendre and Lobatto polynomials and to obtain generalized Gauss-Lobatto points in the element contains a jump of a.

W. Cao, X. Zhang, and Z.Z., Advances Comput. Math. 2017

- LDG for $iu_t + u_{xx} = 0$ (semi-discretization): h^{2k+1} etc.

  L. Zhou, Y. Xu, Z.Z., and W. Cao, JSC 2017

- LDG for $u_t + au_x = 0$ with degenerate variable coefficient (semi-discretization)

  The best possible superconvergence rate:

  $O(h^{k+3/2})$ or $O(h^{k+5/4})$ (if $a'(x) = 0$ as well) etc.

  W. Cao, C.-W. Shu, and Z.Z., ESAIM: M2NA 2017

- LDG for nonlinear problem $u_t + f(u)_x = 0$, $O(h^{2k+1})$ nodal convergent rate

  before the "shock", W. Cao, C.-W. Shu, Y. Yang, and Z.Z., SINUM 2018

- C^1 Petrov-Galerkin method for the two-point boundary value problem

  $O(h^{2(k-1)})$ for both $u - u_h$ and $(u - u_h)'$. W. Cao, L. Jia, and Z.Z. 2021

What's left?

- A rigorous proof: no need to make initial correction for parabolic equations
- Theory for fully discretization in both time and space (Some program)
- Higher-order problems (Some program has been made)
- Non-linear problems (Some program has been made)
- ....

A FVM for the Sturm-Liouville System

$$-(\beta u')' + qu = f, \quad u(0) = 0 = u(1); \quad \beta(x) \geq \beta_0 > 0, \quad q(x) \geq 0.$$

Partition $\mathcal{P}$: $0 = x_0 < x_1 < \cdots < x_n = 1$

On each element $\tau_i = [x_{i-1}, x_i]$, find r Gauss points $\{g_{i,j}\}_{j=1}^r$

Dual mesh $\mathcal{P}'$: $[g_{i,j}, g_{i,j+1}]$, $i = 1, \ldots, n$; $j = 1, \ldots, r$ with $g_{i,r+1} = g_{i+1,1}$, $g_{n,r+1} = 1$, and skip the interval $[0, g_{1,1}]$ due to $u(0) = 0$

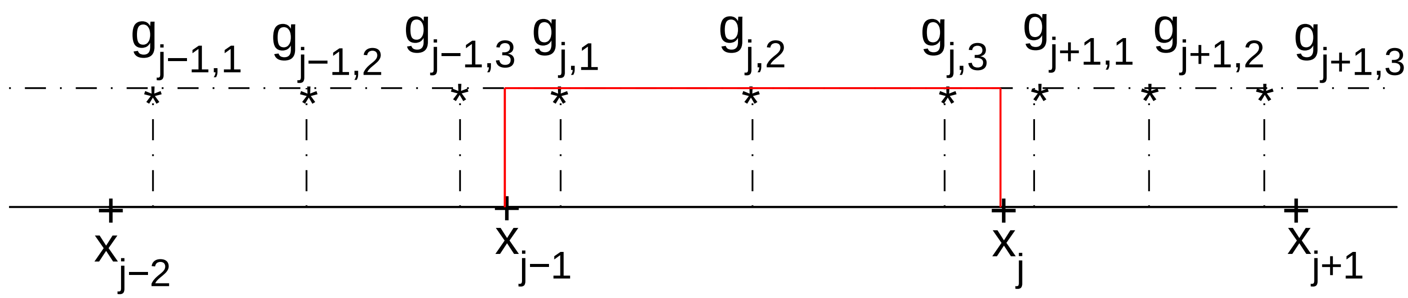


Figure 6: primary and dual meshes

N_i: the piecewise linear nodal shape functions (the "tent" function) at x_i

$\phi_{i,j}$: the Lobatto polynomials (the "bubble" functions) on τ_i.

Our FVM is to find $u_{\mathcal{P}} = \sum_{i=1}^{n-1} c_i N_i + \sum_{i=1}^{n} \sum_{j=2}^{r} c_{i,j}\phi_{i,j}$ such that

$$(\beta u'_{\mathcal{P}})(g_{i,j}) - (\beta u'_{\mathcal{P}})(g_{i,j+1}) + \int_{g_{i,j}}^{g_{i,j+1}} q u_{\mathcal{P}} = \int_{g_{i,j}}^{g_{i,j+1}} f \tag{4}$$

Note: There are $nr-1$ unknows and $nr-1$ equations, we can show that the system is uniquely solvable for c_i's and $c_{i,j}$'s

For analysis, we may re-write (4) into variational format $a(u_{\mathcal{P}}, w) = (f, w)$

Denote $\chi_{i,j}$, the charicteristic function on the subinterval $[g_{i,j}, g_{i,j+1}]$

The test function can be written as $w = \sum_{i=1}^{n} \sum_{j=1}^{r} w_{i,j} \chi_{i,j}$ with $w_{n,r} = 0$

Multiplying both sides of (4) by $w_{i,j}$, summing up, and re-grouping, we have

$$a(u_{\mathcal{P}}, w) = \sum_{i=1}^{n} \sum_{j=1}^{r} (\beta u'_{\mathcal{P}})(g_{i,j})[w_{i,j}] + \int_0^1 q u_{\mathcal{P}} w = \int_0^1 f w$$

Define a dual norm by $|w|^2_{\mathcal{P}'} = \sum_{i=1}^{n} h_i^{-1} \sum_{j=1}^{r} [w_{i,j}]^2$ with $[w_{i,j}] = w_{i,j+1} - w_{i,j}$

Then for any v in the trial space, we can prove (Petrov-Galerkin)

$$\sup_w \frac{a(v, w)}{|w|_{\mathcal{P}'}} \geq C|v|_1, \quad C = \beta_0 \text{ if } q = 0.$$

Key observation

Gauss-type quadrature vs Newton-type quadrature

When $q = 0$, $u \in \mathcal{P}_{2r}$, $\beta u'_{\mathcal{P}}$ is exact at all Gaussian points for variable $\beta(x)$!

Denote $e_{\mathcal{P}} = u - u_{\mathcal{P}}$, we have

$$(\beta e'_{\mathcal{P}})(g_{i,j}) - (\beta e'_{\mathcal{P}})(g_{i,j+1}) = 0$$

and therefore $(\beta e'_{\mathcal{P}})(g_{i,j})$ is constant and we denote it as K.

Let $u \in \mathcal{P}_{2r}$, a polynomial of degree no more than $2r$, then the r-point Gaussian quadrature is exact for the integration of $e'_{\mathcal{P}}$, and hence

$$0 = \int_{-1}^{1} e'_{\mathcal{P}}(x)dx = \sum_{i,j} e'_{\mathcal{P}}(g_{i,j})w_{i,j} = K \sum_{i,j} \frac{w_{i,j}}{\beta(g_{i,j})}.$$

Therefore, $K = 0$,

Superconvergence

- Function values at nodal points:

$$(u - u_{\mathcal{P}})(x_{i+1}) = (u - u_{\mathcal{P}})(x_i) + O(h^{2r+1})$$

$$(u - u_{\mathcal{P}})(x_1) = O(h^{2r+1}), \quad (u - u_{\mathcal{P}})(x_i) = O(h^{2r})$$

- Derivative superconvergence at the Gauss points

$$(u - u_{\mathcal{P}})'(g_{i,j}) = O(h^{r+2}), \quad = O(h^{2r}) \quad \text{when} \quad q = 0$$

- Function value supercovergence at interior Lobatto points:

$$(u - u_{\mathcal{P}})(l_{i,j}) = O(h^{r+2}), \quad r > 1$$

Comparing with FEM

- Nodes: same rate of convergence for function value approximation

  one-order higher at those nodes near x_0 and x_n.

- Lobbato points: same rate of convergence for function value approximation

- Gauss points: better convergent rate for the derivative approximation

$$O(h^{r+2}) \quad \text{comparing with} \quad O(h^{r+1}) \quad \text{in general, and}$$

$$O(h^{2r}) \quad \text{comparing with} \quad O(h^{r+1}) \quad \text{when} \quad q = 0$$

- Note: Numerical tests indicate that our error estimates are sharp!

Waixiang Cao, Zhimin Zhang, and Qingsong Zou, Superconvergence of any order finite volume schemes for 1D general elliptic equations, Journal of Scientific Computing 56 (2013), 566-590.

The p-version

Fix mesh (h), increase polynomial degree $r = p$.

Additional hypothesis: $\|u^{(k)}\|_\infty \leq cM^k$.

Superconvergence results:

$$|u - u_p|_L = \max_{i,j} |(u - u_p)(l_{i,j})| \leq C \left(\frac{heM}{4p}\right)^{p+2} \quad \text{compare} \quad \left(\frac{heM}{4p}\right)^{p+1}$$

$$|u - u_p|_G = \max_{i,j} |(u - u_p)'(g_{i,j})| \leq C \left(\frac{heM}{4p}\right)^{p+1} \quad \text{compare} \quad \left(\frac{heM}{4p}\right)^{p}$$

Waixiang Cao, Zhimin Zhang, and Qingsong Zou, Analysis of a p-version finite volume method for 1-D elliptic problems, Journal of Computational and Applied Mathematics 265 (2014), 17-32.

Numerical test

$\beta = e^x$, $a = 1$, $q = 7$. $u(x) = \sin(4\pi x)$ on $[-1, 1]$

$h = 1$ (two elements), $p = 15, 16, \ldots, 24$.

Convergence rates are realized by ratios

$$\|u - u_p\|_0 : \left(\frac{heM}{4p}\right)^{p+1}, \quad \|u - u_p\|_\infty : \left(\frac{heM}{4p}\right)^{p+1}$$

$$|u - u_p|_L : \left(\frac{heM}{4p}\right)^{p+2}, \quad |u - u_p|_G : \left(\frac{heM}{4p}\right)^{p+1}$$

$$|u - u_p|_1 : \frac{1}{\sqrt{p}} \left(\frac{heM}{4p}\right)^{p}$$

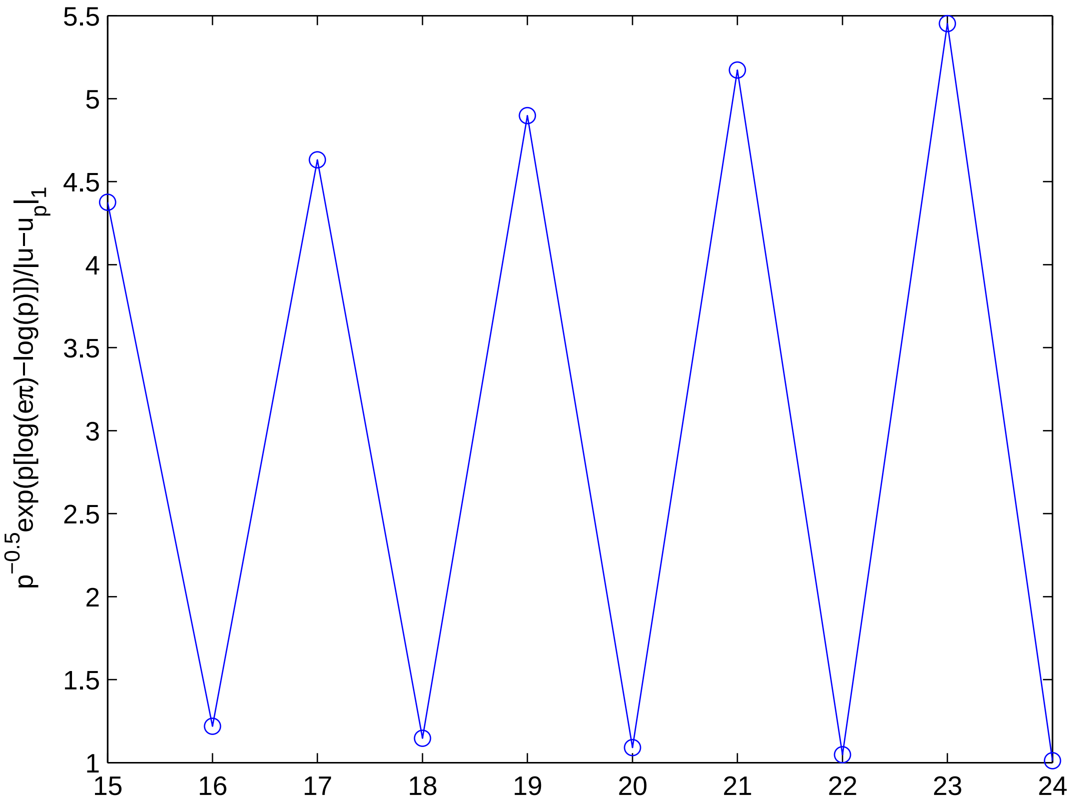


Figure 7: Ratio of estimated error and computed error for $|u - u_p|_1$

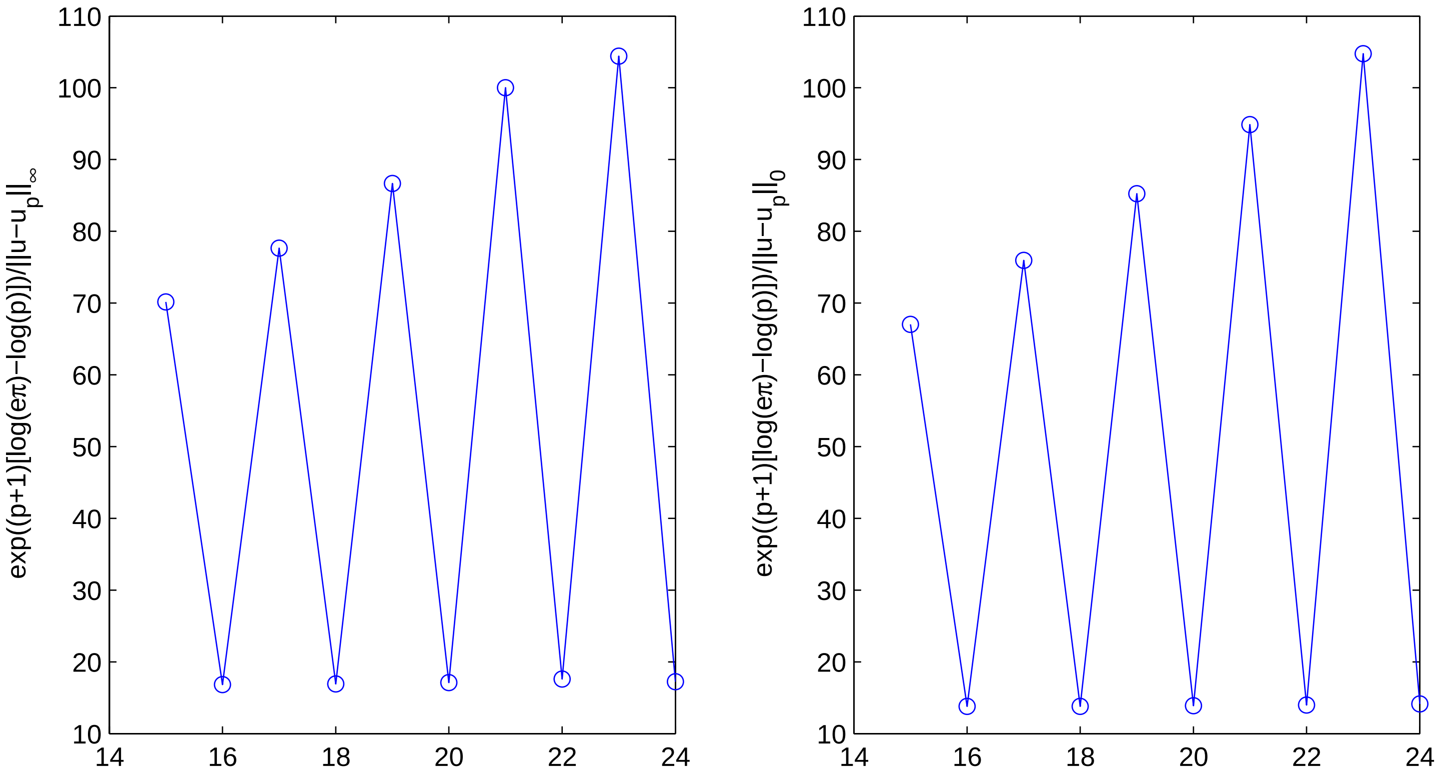


Figure 8: Ratio of estimated error and computed error. $\|u-u_p\|_\infty$ (left), $\|u-u_p\|_0$

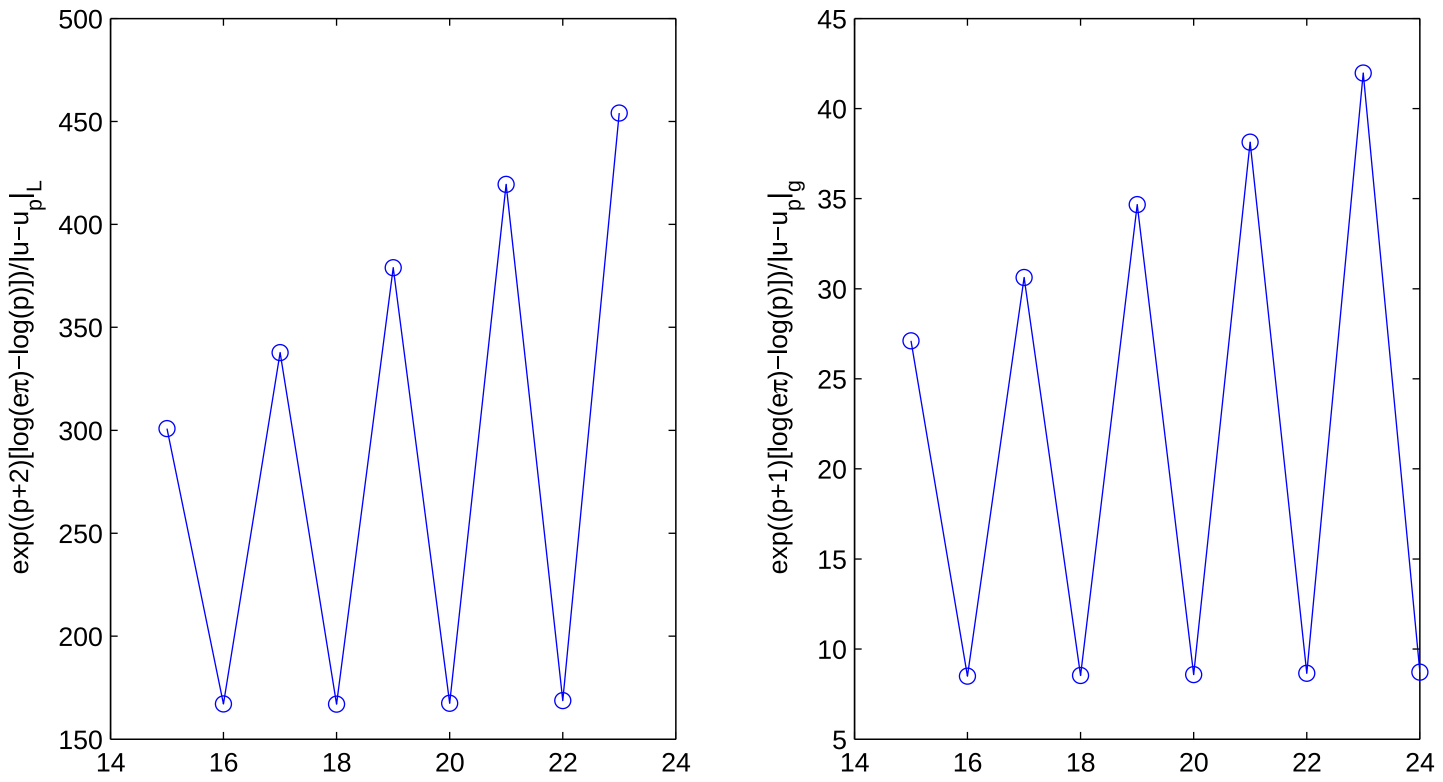


Figure 9: Ratio of estimated error and computed error. Lobatto (left), Gauss

2D: the Poisson equation

$-\Delta u = f$, Homogeneous Dirichlet boundary condition

rectangular elements with tensor product trial space of degree r

- Convergent rate at vertices:

$$(u - u_{\mathcal{P}})(x_i, y_j) \approx O(h^{2r}) \quad \text{(same as FEM)}$$

- Convergent rate at Lobatto-Lobatto points:

$$(u - u_{\mathcal{P}})(l^x_{i,k}, l^y_{j,m}) \approx O(h^{r+2}), \quad k, m = 1, \ldots, r-1 \quad \text{(same as FEM)}$$

- Convergent rate of the gradient at Gauss-Gauss points:

$$\nabla(u - u_{\mathcal{P}})(g^x_{i,k}, g^y_{j,m}) \approx O(h^{r+1}), \quad k, m = 1, \ldots, r \quad \text{(same as FEM)}$$

What is different from FEM? On the element $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$:

$$\partial_x (u - u_{\mathcal{P}})(g^x_{i,k}, l^y_{j,m}) \approx O(h^{r+2}), \quad k = 1, \ldots, r;\; m = 0, 1, \ldots, r.$$

$$\partial_y (u - u_{\mathcal{P}})(l^x_{i,m}, g^y_{j,k}) \approx O(h^{r+2}), \quad m = 0, 1, \ldots, r;\; k = 1, \ldots, r.$$

where $l^y_{j,m}$ is the $m+1$th Lobatto points on the interval $[y_{j-1}, y_j]$.

This convergent rate can be realized by the the following discrete L^2-norm:

$$\sum_{i,j} \sum_{k=1}^{r} \sum_{m=0}^{r} ([\partial_x (u - u_{\mathcal{P}})(g^x_{i,k}, l^y_{j,m})]^2 + [\partial_y (u - u_{\mathcal{P}})(l^x_{i,m}, g^y_{j,k})]^2) w^g_k w^l_m h^x_i h^y_j$$

where w^g_k's are weights for the r-point Gauss quadrature

and w^l_m's are the weights for the $r+1$-point Lobatto quadrature.

Numerical test

Model problem $-\Delta u = f$ on the unit square Ω with

$$\begin{aligned} f &= [(5\pi^2 - 4y^2 - 3)\sin(\pi x)\sin(2\pi y) - 2\pi\cos(\pi x)\sin(2\pi y) \\ &\quad -8\pi y\sin(\pi x)\cos(2\pi y)]e^{x-0.5}e^{y^2}. \end{aligned}$$

The exact solution is $u(x,y) = \sin(\pi x)e^{x-0.5}\sin(2\pi y)e^{y^2}$. Ω is divided into $N \times N$ sub-squares with $N = 2, 4, \ldots, 64$. Denote $\mathcal{N}_h$, $\mathcal{N}^g$, and $\mathcal{N}^l$ as sets of vertices, Gauss points, and Lobatto points, respectively. Define

$$e_G = \max_{Q\in\mathcal{N}^g} |\nabla(u - u_h)(Q)|, \quad e_L = \max_{P\in\mathcal{N}^l} |(u - u_h)(P)|$$

$$e_N = \max_{P\in\mathcal{N}_h} |(u - u_h)(P)|.$$

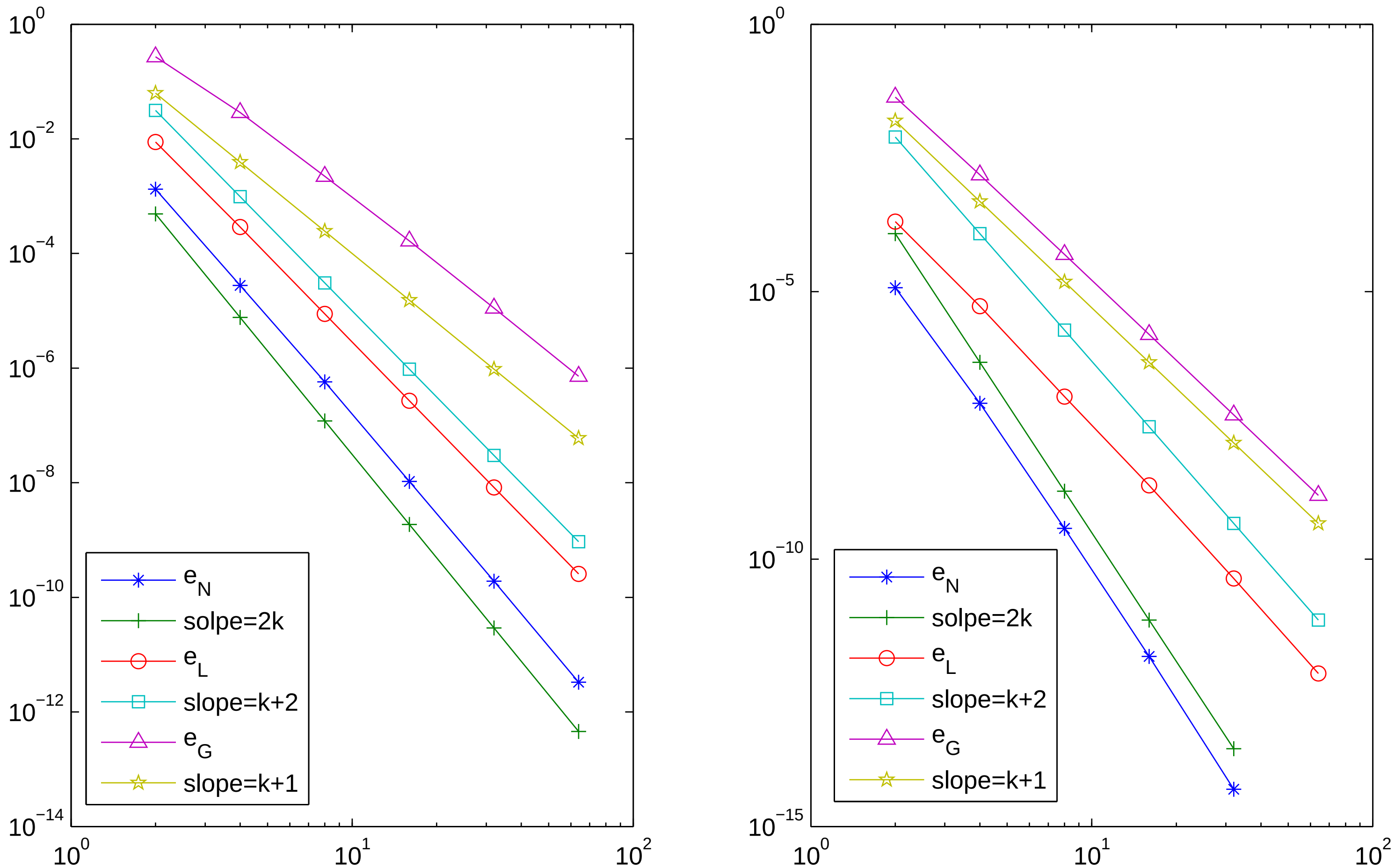


Figure 10: left: $k = 3$, right: $k = 4$.

What we have done in theory for 2D Poisson equation?

- Reformulate the FVM to a Petrov-Galerkin FEM
- A general approach to establish the inf-sup condition for any r
- Optimal rate of convergence in both H^1 and L^2 norms
- Superconvergence to u at vertices, $O(h^{2r})$ vs $O(h^{r+1})$
- Superconvergence to u at Lobitto-Lobatto points, $O(h^{r+2})$ vs $O(h^{r+1})$
- Supercovergence to ∇u at Gauss-Gauss points, $O(h^{r+1})$ vs $O(h^r)$

Broader Open Problems

- Systematic superconvergence theory for FVM and DG

(including nonconforming) methods under triangular meshes

- Gradient recovery for nonconforming FEM, FVM, DG, ...

- Q: Is symmetry theory valid for FVM and/or DG?

A: Probably not for DG, especially LDG.

- Q: Is the computer-based proof valid for FVM and DG?

A: My answer is yes, and much involved work is necessary.

- Applications such as PDE eigenvalue problems ....

Note I did not discuss the Virtual Element Method (VEM) in this talk.

Thank you!

Zhimin Zhang and Qingsong Zou, Some recent advances on vertex centered finite volume element methods for elliptic equations, *Science China, Mathematics*, Special Issue on Computational Mathematics, Guest Editors: Zhiming Chen, Weinan E, and Chi-Wang Shu, Vol.56 No.12 (2013), 2507-2522.

Waixiang Cao, Zhimin Zhang, and Qingsong Zou, Superconvergence of any order finite volume schemes for 1D singularly perturbed problems, *Journal of Computational Mathematics* 31-5 (2013), 488-508.

Waixiang Cao, Zhimin Zhang, and Qingsong Zou, Is $2k$-Conjecture valid for finite volume methods? *SIAM Journal on Numerical Analysis* 53-2 (2015), 942-962.

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