Stabilized Finite Element Methods for the Navier-Stokes Equations..

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[1] B. G-A, V. John, and J. Novo. On the Convergence Order of the Finite Element Error in the Kinetic Energy for High Reynolds Number Incompressible Flows. *CMAME.*, (submitted)

Outline

1) Introduction

Numerical difficulties in convection-dominated problems

2) Discretization by F-E methods.

3) Convection diffusion equation: Analysis

The effect of convection: order (of convergence) reduction by 1

Stabilized methods: order reduction by 1/2

4) Navier-Stokes equations

The effect incompressibilty: order (of convergence) reduction by 2

Stabilized methods: order reduction by 1

Stabilized methods: order reduction by 1/2

5) References

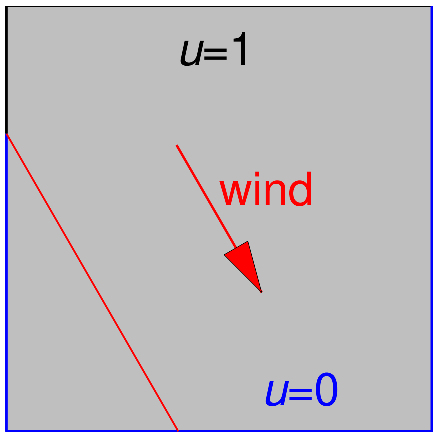
Introduction

- Stardard numerical methods perform poorly in convection-dominated problems (spurious oscillations if boundary or internal layers present).

Example

$$\left.\begin{array}{ll} -\varepsilon\Delta u + b\cdot\nabla u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_0, \\ u = 1 & \text{on } \Gamma_1, \end{array}\right\}$$

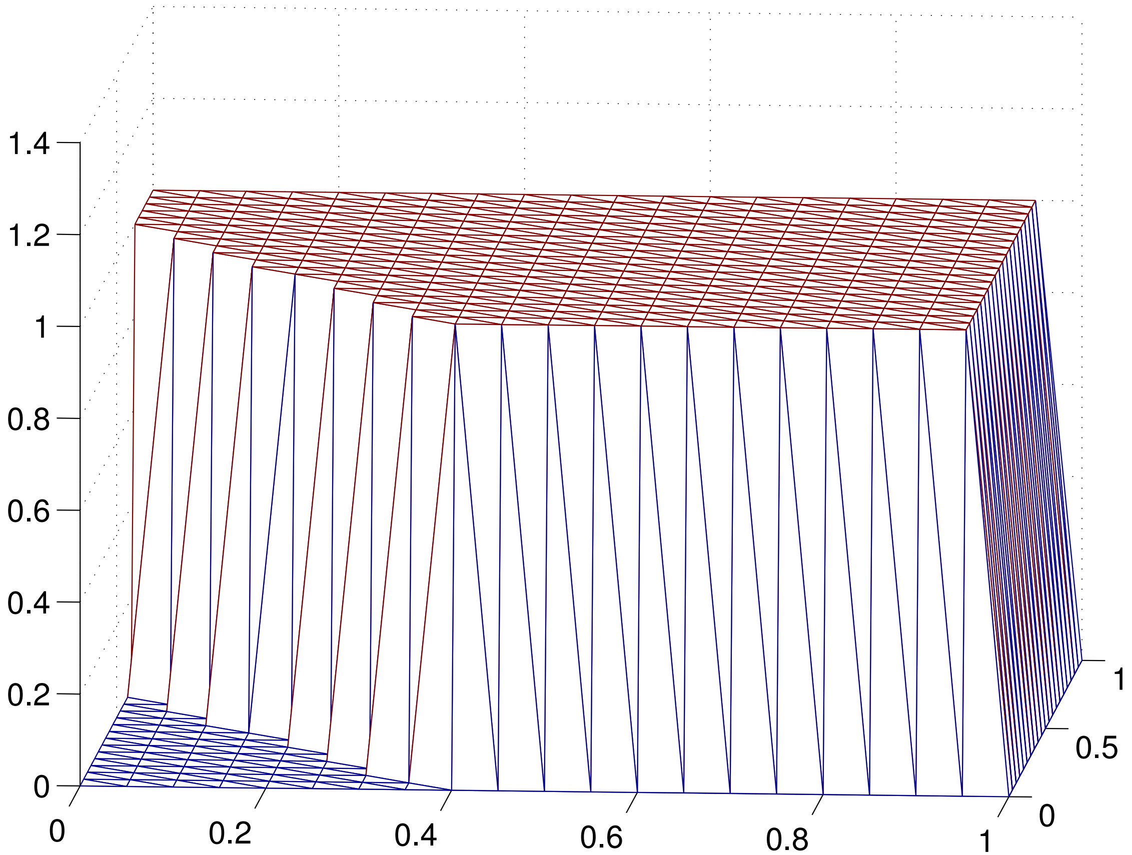
where $\Omega = [0,1]\times[0,1], \quad b = [\cos(\pi/3), -\sin(\pi/3)]^T, \quad f(x) = 0.$



Internal layer

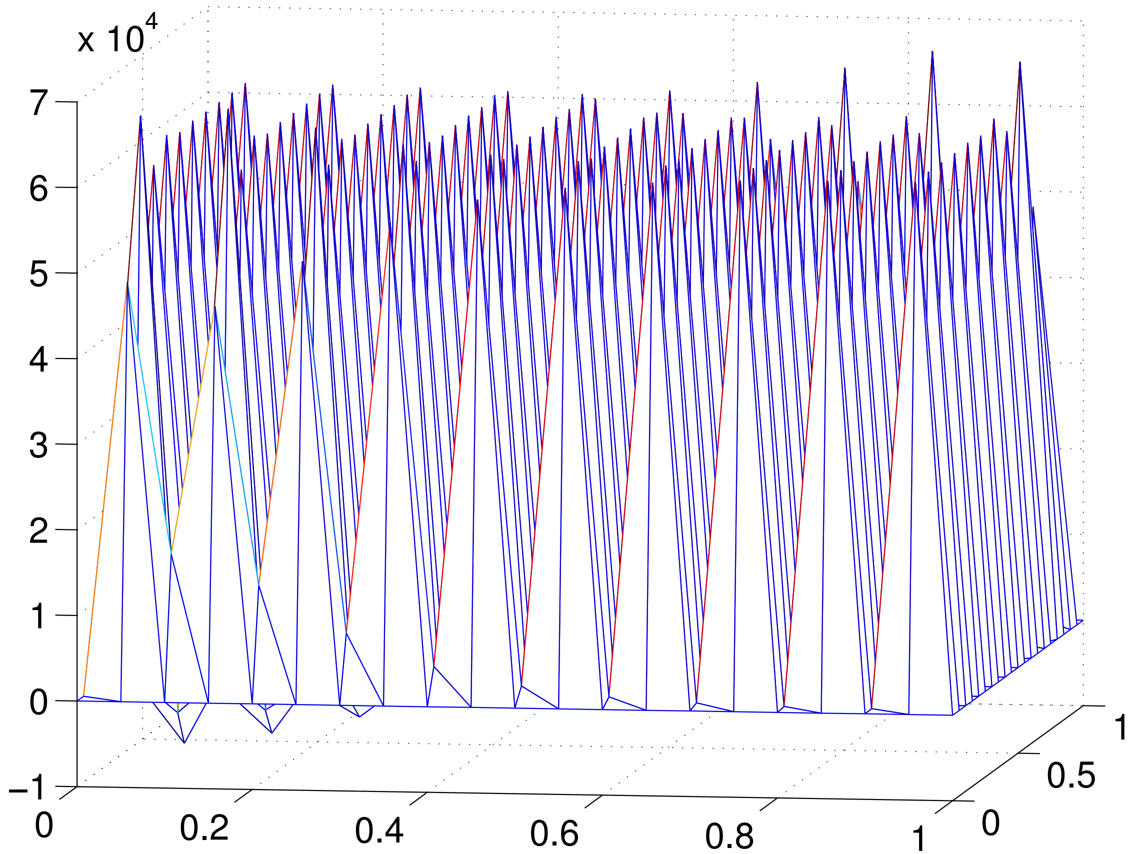
boundary layer

Example



The true solution for $\epsilon = 10^{-8}$ interpolated on the grid

Example



Standard Galerkin approximation.

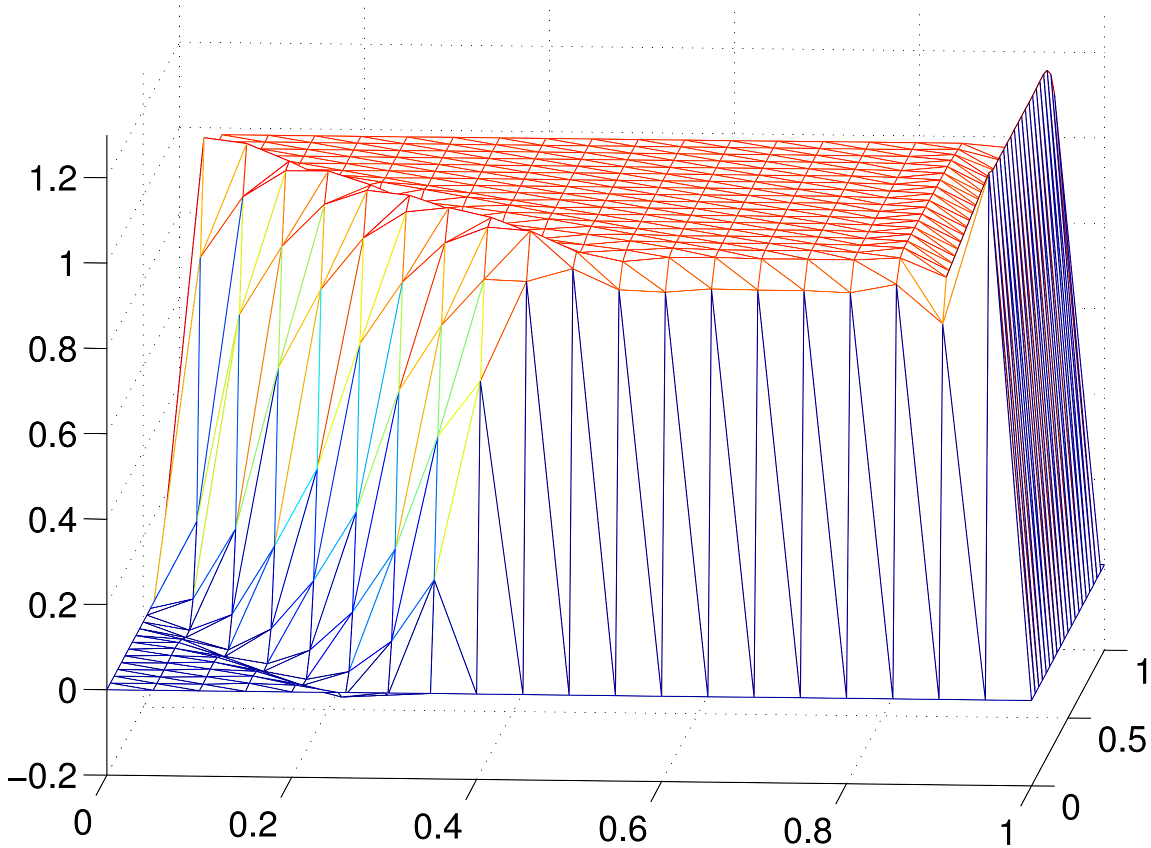
Introduction

- Stardard numerical methods perform poorly in convection-dominated problems (spurious oscillations if boundary or internal layers present).

- Stabilized methods perform better. But yet ...

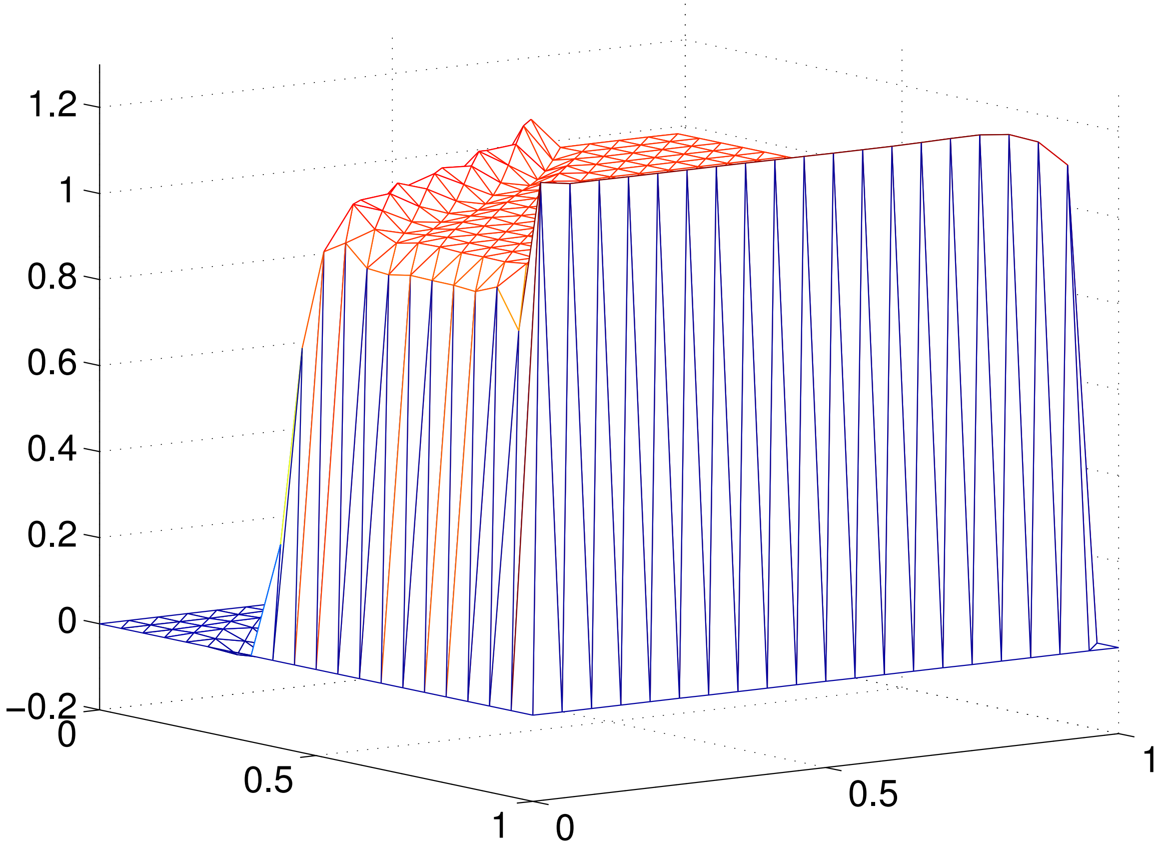
  Stabilized methods: Standard methods with extra terms that vanish on the true solution (but not on the numerical approximation) and improve the quality and/or accuracy of the approximation.

Example



Streamline diffusion (SUPG) approximation.

Example



Streamline diffusion (SUPG) approximation from a different point of view.

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- And even for not so convection-dominated problems

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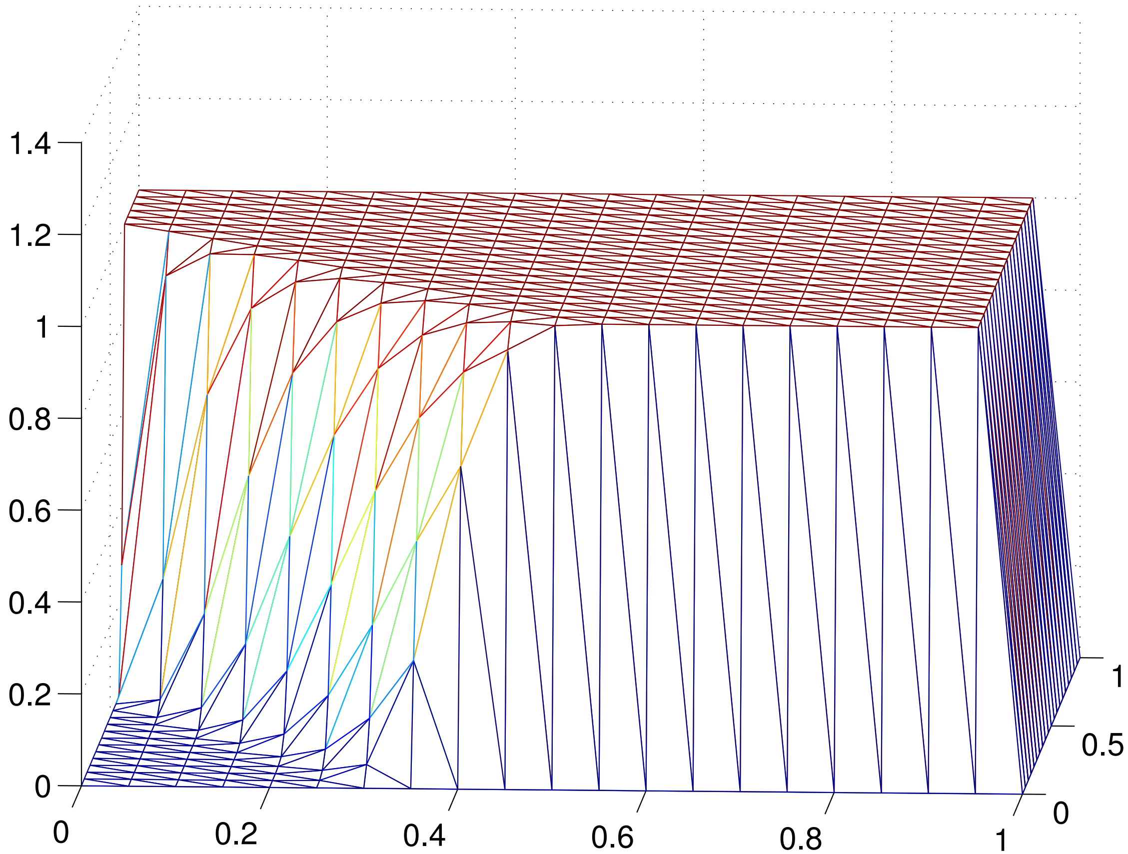
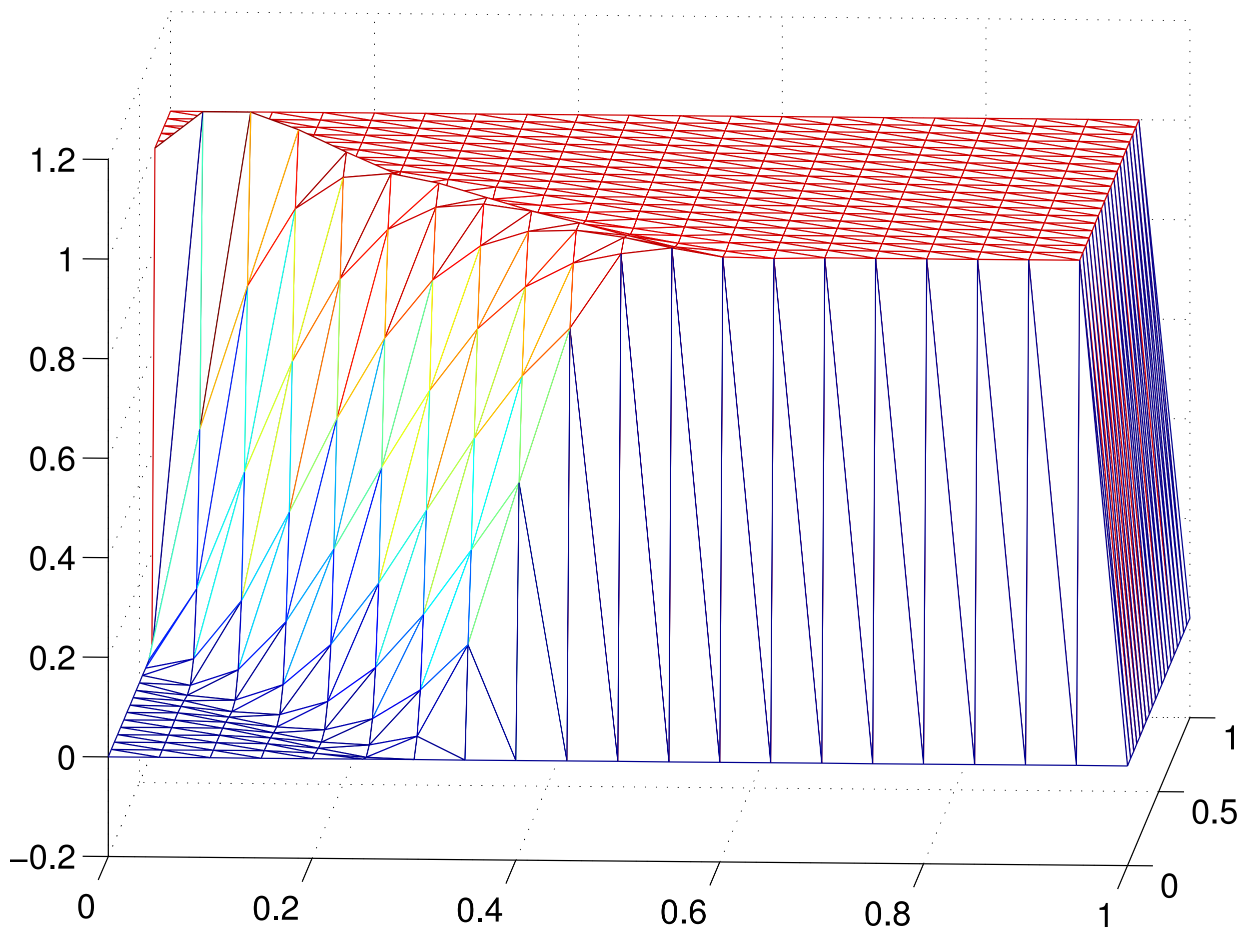


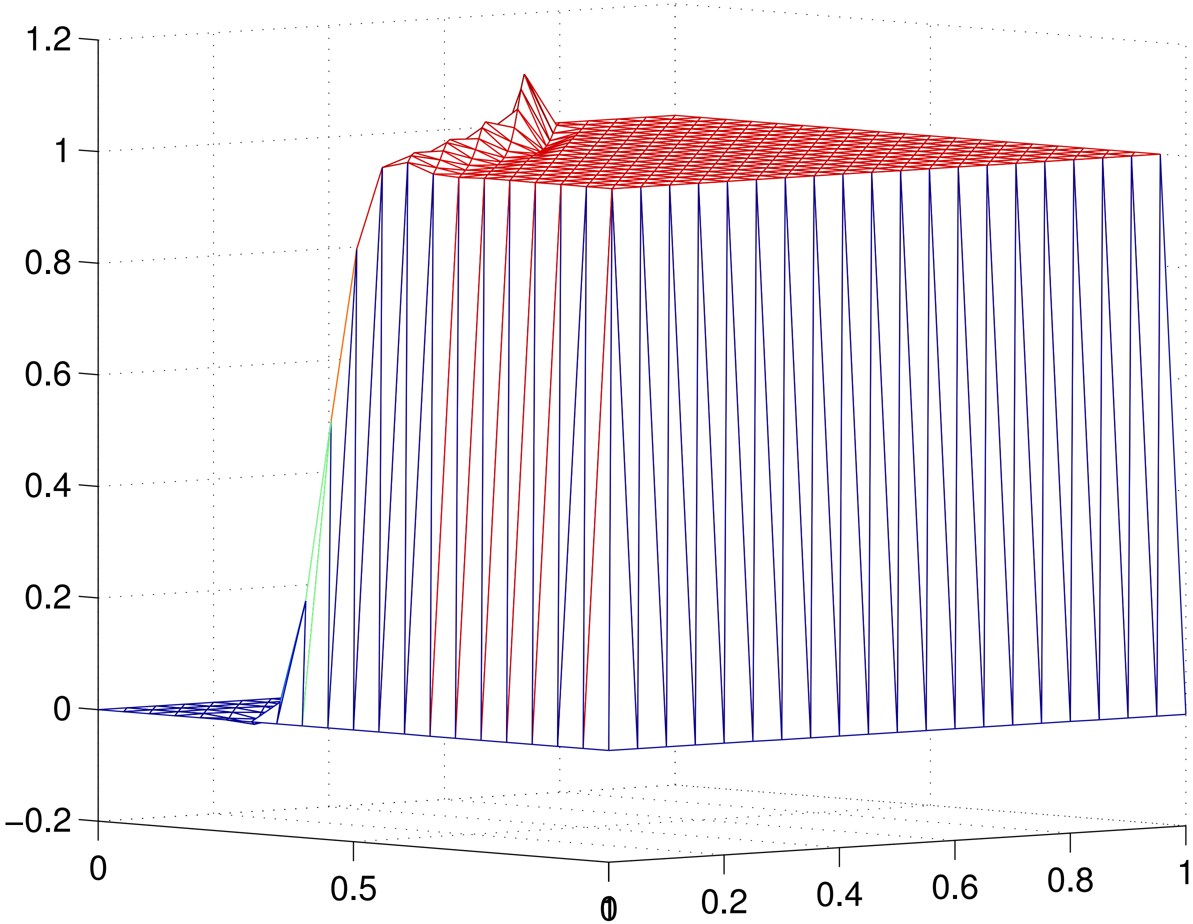
The true solution for $\epsilon = 10^{-3}$ interpolated on the grid

Example



SUPG-A (by Lube) approximation for $\epsilon = 10^{-3}$.

Example

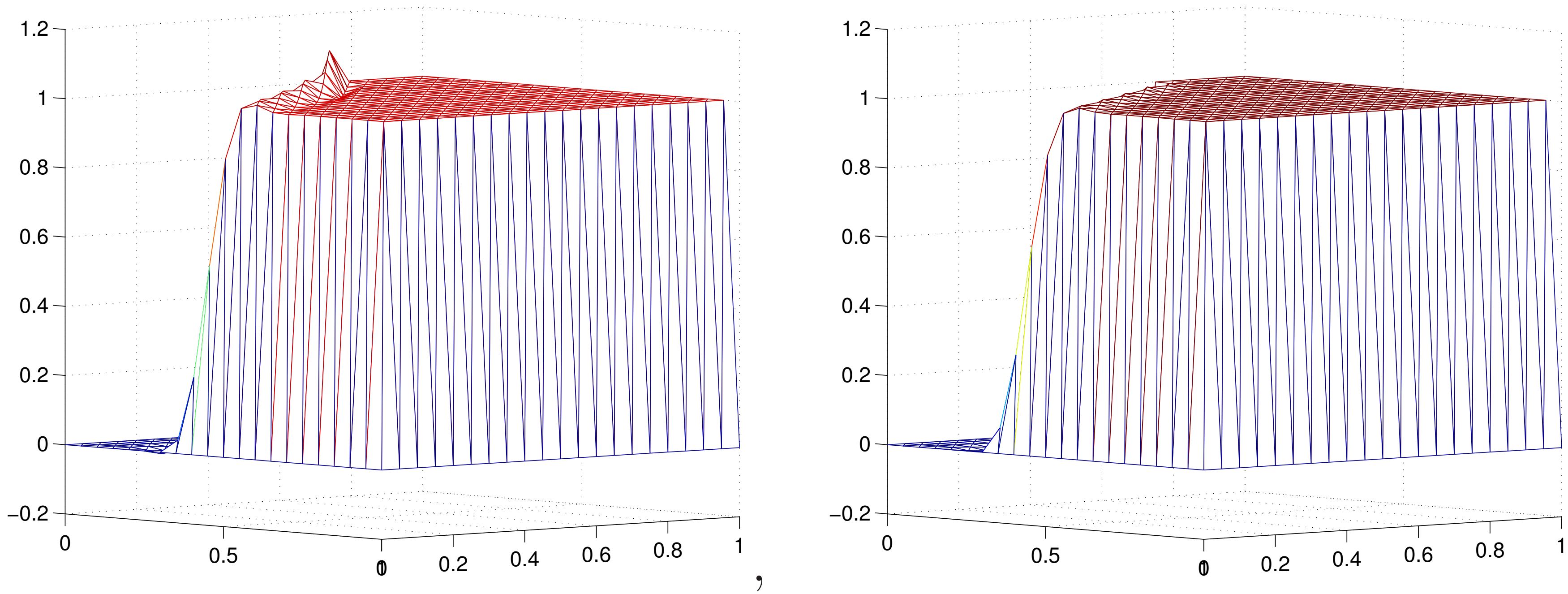


SUPG-A approximation for $\epsilon = 10^{-3}$ from different point of view .

Introduction

- Stardard numerical methods perform poorly in convection-dominated problems (spurious oscillations if boundary or internal layers present).
- Stabilized methods perform better. But yet ...
- And even for not so convection-dominated problems
- Furhter not-well-understood techniques have to be applied to obtain accurate approximations.

Example



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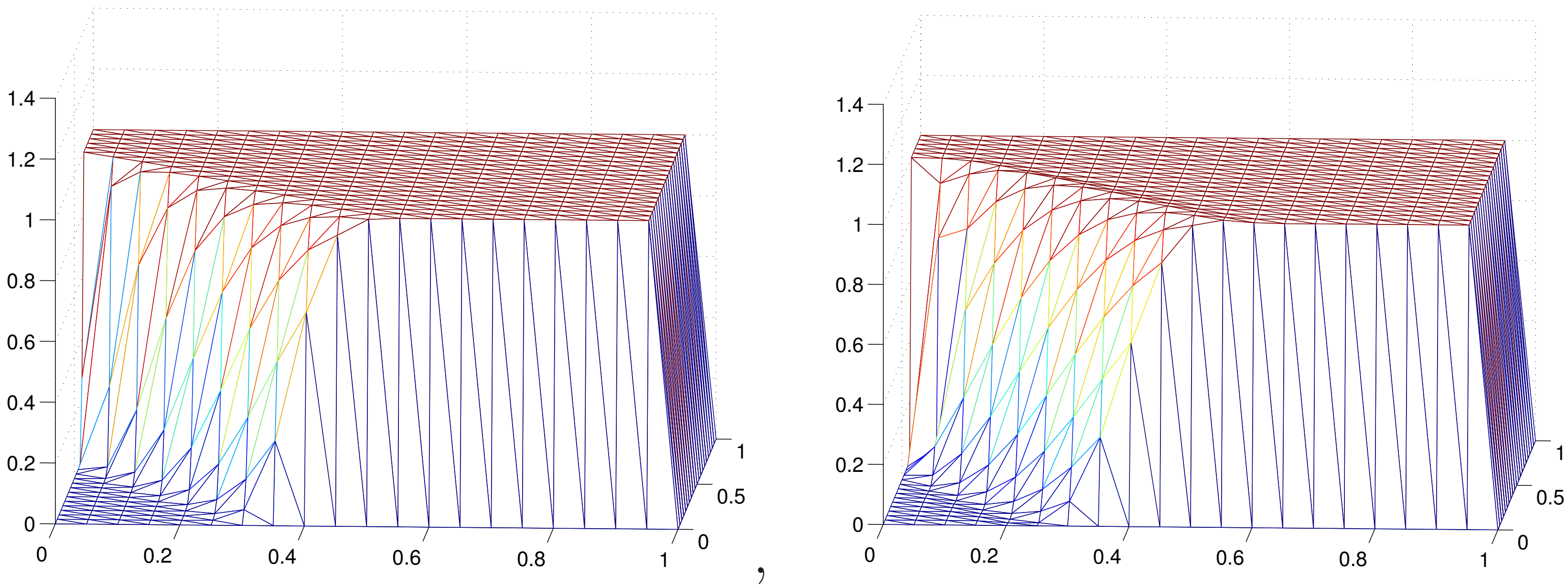
The SUPG-A (left) and the (shortened) SMS (right)
approximations for $\epsilon = 10^{-3}$

Example



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The interpolant of the true solution (left) and the (shortened) SMS approximation (right) for $\epsilon = 10^{-3}$

Introduction

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- Stabilized methods perform better. But yet ...
- And even for not so convection-dominated problems
- Furhter not-well-understood techniques have to be applied to obtain accurate approximations.
- A key ingredient for good numerical results seem to be error bounds valid for vanishing diffusion (i.e., independent of the inverse of the diffusion parameter).

Introduction

- Stardard numerical methods perform poorly in convection-dominated problems (spurious oscillations if boundary or internal layers present).
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- And even for not so convection-dominated problems
- Furhter not-well-understood techniques have to be applied to obtain accurate approximations.
- A key ingredient for good numerical results seem to be error bounds valid for vanishing diffusion (i.e., independent of the inverse of the diffusion parameter).
- Analysis in time-dependent problem much less developed than in steady problems.

The Navier-Stokes equations

$$
\begin{aligned}
\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } (0, T] \times \Omega, \\
\nabla \cdot \mathbf{u} &= 0, \quad \text{in } (0, T] \times \Omega, \\
\mathbf{u}(0, \cdot) &= \mathbf{u}_0(\cdot) \quad \text{in } \Omega, \\
\mathbf{u} &= 0, \quad \text{on } (0, T] \times \partial\Omega.
\end{aligned}
$$

$$
\Omega \subset \mathbb{R}^d, \qquad (d = 2, 3),
$$

$$
\nu \ll 1, \qquad \mathrm{Re} = \frac{u_c l_c}{\nu} \gg 1.
$$

Clarification

In R^d, $d = 2, 3$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_d \end{bmatrix},$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \begin{bmatrix} \mathbf{u} \cdot \nabla u_1 \\ \vdots \\ \mathbf{u} \cdot \nabla u_d \end{bmatrix}.$$

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Convection-difusssion equation

$$\begin{array}{rcll} u_t - \nu\Delta u + \mathbf{b}\cdot\nabla u + cu & = & f & \text{in } (0,T]\times\Omega, \\ u(0,\cdot) & = & u_0(\cdot) & \text{in } \Omega, \\ u & = & 0 & \text{on } [0,T]\times\partial\Omega, \end{array}$$

$$\Omega \subset \mathbb{R}^d, \qquad (d=2,3),$$

$$\mathbf{b} = \mathbf{b}(t,\mathbf{x}), \quad c = c(t,x)),$$

$$\nu \ll 1, \qquad \mathrm{Pe} = \frac{\|\mathbf{b}\|_\infty l_c}{\nu} \gg 1.$$

Technical assumption: $\quad 0 < \mu_0 \leq c - \frac{1}{2}\nabla\cdot\mathbf{b} \leq \mu_1.$

The Navier-Stokes equations

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$$

Weak form: find $(\mathbf{u},p) : [0,T] \to H_0^1(\Omega)^d \times L_0^2(\Omega)$ satisfying $\mathbf{u}(0) = \mathbf{u}_0$, and

$$
\begin{aligned}
(\partial_t\mathbf{u},\boldsymbol{\varphi}) + \nu(\nabla\mathbf{u},\nabla\boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u},\boldsymbol{\varphi}) - (p,\nabla\cdot\boldsymbol{\varphi}) = (\boldsymbol{f},\boldsymbol{\varphi})), &\quad \boldsymbol{\varphi}\in H_0^1(\Omega)^d,\\
(\nabla\cdot\mathbf{u},\psi) = 0, &\quad \psi\in L_0^2(\Omega),
\end{aligned}
$$

where $\quad L_0^2(\Omega) = \{q\in L^2(\Omega) : (q,1) = 0\}, \qquad (\cdot,\cdot)$ inner prod. in $L^2(\Omega)$.

Clarification

In R^d, $d = 2, 3$, $\quad (\cdot, \cdot)$ inner prod. in $L^2(\Omega)$.

$$(v, w) = \int_\Omega vw \, dx_1 \ldots \, dx_d,$$

$$(\mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{v} \cdot \mathbf{w} \, dx_1 \ldots \, dx_d,$$

The Navier-Stokes equations

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Weak form: find $u : [0,T] \to H_0^1(\Omega)$ satisfying $u(0) = u_0$, and

$$(\partial_t u, \varphi) + \nu(\nabla u, \nabla\varphi) + (\mathbf{b}\cdot\nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \quad \varphi \in H_0^1(\Omega),$$

Convection-diffusion eqn.: finite element discretization

- $\mathcal{T}_h$ partition of Ω into elements of max. size h
- $V_h \subset H_0^1(\Omega)$ (piecewise polynomials of degree k).

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Galerkin discretization: find $u_h : [0,T] \to V_h$ satisfying $u_h(0) \approx u_0$, and

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla\varphi_h) + (\mathbf{b}\cdot\nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \varphi_h \in V_h.$$

Navier-Stokes eqns.: mixed finite element discretization

- $\mathcal{T}_h$ partition of Ω into elements of max. size h
- $\mathbf{V}_h \subset H_0^1(\Omega)^d$ and $Q_h \subset L_0^2(\Omega)$ satisfying

$$\inf_{q_h \in Q_h} \sup_{v_h \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)}{\|\nabla \mathbf{v}_h\| \|q_h\|} \geq \beta_0.$$

where $\| \cdot \|$ norm in $L^2(\Omega)$ (associated with $(\cdot, \cdot)$).

Also, $\|\cdot\|_l$ norm in Sobolev's space $H^l(\Omega)$.

And, $\|\cdot\|_\infty$ norm in $L^\infty(\Omega)$.

The Navier-Stokes equations: Discretization

Find $(\mathbf{u}, p) : [0, T] \to H_0^1(\Omega) \times L_0^2(\Omega)$ satisfying $\mathbf{u}(0) = u_0$, and

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,$$
$$(\nabla \cdot \mathbf{u}, \psi) = 0, \quad \psi \in L_0^2(\Omega),$$

Galerkin Discretization:

Find $(\mathbf{u}_h, p_h) : [0, T] \to \mathbf{V}_h \times Q_h$ satisfying $\mathbf{u}_h(0) \approx \mathbf{u}_0$, and

$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla \cdot \boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h),$$
$$\boldsymbol{\varphi}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \psi_h) = 0, \quad \psi_h \in Q_h,$$

where $$B(\mathbf{u}, \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + \tfrac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in H_0^1(\Omega)^d$$

Basics of the error analysis

C-V equations: compare approximation u_h with elliptic projection $\pi_h u \in V_h$ defined by

$$(\nabla \pi_h u, \nabla \varphi_h) = (\nabla u, \nabla \varphi_h), \quad \forall \varphi_h \in V_h.$$

and satisfying that for $u \in H_0^1(\Omega) \cap H^{r+1}(\Omega)$,

$$\|u - \pi_h u\| + h\|\nabla(u - \pi_h u)\| \leq C h^{k+1} \|u\|_{k+1}.$$

N-S equations: compare approximation $\mathbf{u}_h$ with Stokes projection $\boldsymbol{s}_h \in \mathbf{V}_h$ (and $l_h \in Q_h$) defined by

$$\begin{array}{rcll} \nu(\nabla \boldsymbol{s}_h, \nabla \boldsymbol{\varphi}_h) - (l_h, \nabla \cdot \boldsymbol{\varphi}_h) & = & \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}_h), & \forall\ \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\ (\nabla \cdot \boldsymbol{s}_h, \psi_h) & = & 0, & \forall \psi_h \in Q_h. \end{array}$$

satisfying that for $\mathbf{u} \in H_0^1(\Omega)^d \cap H^{k+1}(\Omega)^d$, with $\nabla \cdot \mathbf{u} = 0$,

$$\|\mathbf{u} - \boldsymbol{s}_h\| + h\|\nabla(\mathbf{u} - \boldsymbol{s}_h)\| + \frac{h}{\nu}\|l_h\| \leq C h^{k+1} \|\mathbf{u}\|_{k+1}.$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

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for $v_h = \pi_h u$

$$\nu(\nabla v_h, \nabla\varphi_h) \qquad = \nu(\nabla u, \nabla\varphi_h)$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } v_h = \pi_h u$$

$$\nu(\nabla v_h, \nabla \varphi_h) \qquad = (f, \varphi_h)$$
$$- (u_t, \varphi_h) - (cu, \varphi_h)$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $v_h = \pi_h u$

$$\begin{aligned}(\partial_t v_h, \varphi_h) + \nu(\nabla v_h, \nabla \varphi_h) \qquad &+ (cv_h, \varphi_h) = (f, \varphi_h) \\ &+ (\partial_t(v_h - u), \varphi_h) - (c(v_h - u), \varphi_h),\end{aligned}$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

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for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, \varphi_h) + \nu(\nabla e_h, \nabla \varphi_h) \qquad + (ce_h, \varphi_h) =$$
$$(\partial_t \varepsilon_h, \varphi_h) + (c\varepsilon_h, \varphi_h),$$

$$(\varepsilon_h = \pi_h u - u),$$

recall $\qquad \|\varepsilon_h\| \le Ch^{k+1}\|u\|_{k+1}, \qquad \|\partial_t \varepsilon_h\| \le Ch^{k+1}\|u_t\|_{k+1},$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

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take $\varphi_h = e_h$,

Analysis: Convection-diffusion eqn.

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$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h),$$

$$\frac{d}{dt}\frac{1}{2}\|e_h\|^2 + \nu\|\nabla e_h\|^2 \qquad + (ce_h, e_h) =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h),$$

Detail

$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h)$$

Applying Cauchy-Schwartz inequality

$$\begin{aligned}(c\varepsilon_h, e_h) = (\sqrt{2}c^{1/2}\varepsilon_h, \frac{c^{1/2}}{\sqrt{2}}e_h) &\leq \left\|c^{1/2}\varepsilon_h\right\|^2 + \frac{1}{4}\left\|c^{1/2}e_h\right\|^2 \\ &= \left\|c^{1/2}\varepsilon_h\right\|^2 + \frac{1}{4}(ce_h, e_h),\end{aligned}$$

$$(\partial_t \varepsilon_h, e_h) = (\frac{\sqrt{2}}{c^{1/2}}\partial_t \varepsilon_h, \frac{c^{1/2}}{\sqrt{2}}e_h) \leq \left\|c^{-1/2}\partial_t \varepsilon_h\right\|^2 + \frac{1}{4}(ce_h, e_h).$$

Detail

$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) \leq \left\|c^{1/2}\varepsilon_h\right\|^2 + \left\|c^{-1/2}\partial_t \varepsilon_h\right\|^2 + \frac{1}{2}(ce_h, e_h).$$

Applying Cauchy-Schwartz inequality

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$$(\partial_t \varepsilon_h, e_h) = (\frac{\sqrt{2}}{c^{1/2}}\partial_t \varepsilon_h, \frac{c^{1/2}}{\sqrt{2}}e_h) \leq \left\|c^{-1/2}\partial_t \varepsilon_h\right\|^2 + \frac{1}{4}(ce_h, e_h).$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) + (ce_h, e_h) = (\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h),$$

$$\frac{d}{dt}\frac{1}{2}\|e_h\|^2 + \nu\|\nabla e_h\|^2 + (ce_h, e_h) \leq \|c^{-1/2}\partial_t \varepsilon_h\|^2 + \|c^{1/2}\varepsilon_h\|^2 + \frac{1}{2}(ce_h, e_h),$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + (ce_h, e_h) =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h),$$

$$\frac{d}{dt}\frac{1}{2}\|e_h\|^2 + \nu\|\nabla e_h\|^2 \qquad + \frac{1}{2}(ce_h, e_h) \leq$$
$$\|c^{-1/2}\partial_t \varepsilon_h\|^2 + \|c^{1/2}\varepsilon_h\|^2$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + (ce_h, e_h) =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h),$$

$$\frac{d}{dt}\frac{1}{2}\|e_h\|^2 + \nu\|\nabla e_h\|^2 \qquad + \frac{\mu_0}{2}\|e_h\|^2 \leq$$
$$\mu_0^{-1}\|\partial_t \varepsilon_h\|^2 + \mu_1\|\varepsilon_h\|^2 \qquad ,$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } e_h = \pi_h u - u_h$$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + (ce_h, e_h) =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h),$$

$$\frac{d}{dt}\frac{1}{2}\|e_h\|^2 \qquad + \frac{\mu_0}{2}\|e_h\|^2 \le$$
$$\mu_0^{-1}\|\partial_t \varepsilon_h\|^2 + \mu_1\|\varepsilon_h\|^2 \qquad ,$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$\|e_h(t)\|^2 \leq e^{-\mu_0 t}\|e_h(0)\|^2 + \frac{2}{\mu_0}\int_0^t \|\partial_t \varepsilon_h\|^2 \, dt + 2\mu_1 \int_0^t \|\varepsilon_h\|^2 \, dt.$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) \qquad + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) \qquad + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$\|e_h(t)\|^2 \leq e^{-\mu_0 t}\|e_h(0)\|^2 + \underbrace{\frac{2}{\mu_0}\int_0^t \|\partial_t \varepsilon_h\|^2 \, dt + 2\mu_1 \int_0^t \|\varepsilon_h\|^2 \, dt}_{O(h^{2(k+1)})}.$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } e_h = \pi_h u - u_h$$

$$(\partial_t e_h, \varphi_h) + \nu(\nabla e_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla e_h, \varphi_h) + (ce_h, \varphi_h) =$$

$$(\partial_t \varepsilon_h, \varphi_h) + (c\varepsilon_h, \varphi_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, \varphi_h),$$

$$(\varepsilon_h = \pi_h u - u),$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, \varphi_h) + \nu(\nabla e_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla e_h, \varphi_h) + (ce_h, \varphi_h) =$$
$$(\partial_t \varepsilon_h, \varphi_h) + (c\varepsilon_h, \varphi_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, \varphi_h),$$
$$(\varepsilon_h = \pi_h u - u),$$

$$\|\varepsilon_h\| \le Ch^{k+1}\|u\|_{k+1}, \quad \|\partial_t \varepsilon_h\| \le Ch^{k+1}\|u_t\|_{k+1}, \quad \|\nabla \varepsilon_h\| \le Ch^{k}\|u\|_{k+1}.$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, \varphi_h) + \nu(\nabla e_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla e_h, \varphi_h) + (ce_h, \varphi_h) =$$
$$(\partial_t \varepsilon_h, \varphi_h) + (c\varepsilon_h, \varphi_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, \varphi_h),$$

$$(\varepsilon_h = \pi_h u - u),$$

take $\varphi_h = e_h$,

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) + (\mathbf{b} \cdot \nabla e_h, e_h) + (ce_h, e_h) =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) + (\mathbf{b} \cdot \nabla e_h, e_h) + (ce_h, e_h) =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

$$(\mathbf{b} \cdot \nabla e_h, e_h) + (ce_h, e_h) = -\frac{1}{2}((\nabla \cdot \mathbf{b})e_h, e_h) + (ce_h, e_h) \geq \mu_0 \|e_h\|^2.$$

$$(0 < \mu_0 \leq c - (\nabla \cdot \mathbf{b})/2 \leq \mu_1)$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } e_h = \pi_h u - u_h$$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + \mu_0 \|e_h\|^2 \le$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h)$$
$$\le \frac{2}{\mu_0}\|\partial_t \varepsilon_h\|^2 + 2\frac{\mu_1^2}{\mu_0}\|\varepsilon_h\|^2 + \frac{1}{\mu_0}\|\mathbf{b} \cdot \nabla \varepsilon_h\|^2 + \frac{\mu_0}{2}\|e_h\|^2$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } e_h = \pi_h u - u_h$$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + \mu_0 \|e_h\|^2 \le$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

$$\frac{d}{dt}\frac{1}{2}\|e_h\|^2 + \nu\|\nabla e_h\|^2 \qquad + \frac{\mu_0}{2}\|e_h\|^2 \le$$
$$\frac{2}{\mu_0}\|\partial_t \varepsilon_h\|^2 + 2\frac{\mu_1^2}{\mu_0}\|\varepsilon_h\|^2 + \frac{1}{\mu_0}\|\mathbf{b} \cdot \nabla \varepsilon_h\|^2,$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } e_h = \pi_h u - u_h$$

$$\|e_h(t)\|^2 \leq e^{-\mu_0 t}\|e_h(0)\|^2 + \frac{4}{\mu_0}\int_0^t \left(\|\partial_t \varepsilon_h\|^2 + \mu_1^2\|\varepsilon_h\|^2\right)\, dt + \frac{2}{\mu_0}\int_0^t \|\mathbf{b} \cdot \nabla \varepsilon_h\|^2\, dt$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$\|e_h(t)\|^2 \leq e^{-\mu_0 t}\|e_h(0)\|^2 + \underbrace{\frac{4}{\mu_0}\int_0^t \left(\|\partial_t \varepsilon_h\|^2 + \mu_1^2\|\varepsilon_h\|^2\right)\, dt}_{O(h^{2(k+1)})} + \underbrace{\frac{2}{\mu_0}\int_0^t \|\mathbf{b} \cdot \nabla \varepsilon_h\|^2\, dt}_{O(h^{2k})}$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + \mu_0 \|e_h\|^2 =$$

$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

$$(\mathbf{b} \cdot \nabla \varepsilon_h, e_h) = -(\varepsilon_h, \mathbf{b} \cdot \nabla e_h) - ((\nabla \cdot \mathbf{b})\varepsilon_h, e_h)$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

$$\text{for } e_h = \pi_h u - u_h$$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad + \mu_0 \|e_h\|^2 =$$
$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

$$(\mathbf{b} \cdot \nabla \varepsilon_h, e_h) = -(\varepsilon_h, \mathbf{b} \cdot \nabla e_h) - ((\nabla \cdot \mathbf{b})\varepsilon_h, e_h)$$
$$\leq \frac{\nu}{2}\|\nabla e_h\|^2 + \frac{\|\mathbf{b}\|_\infty^2}{2\nu}\|\varepsilon_h\|^2 + \frac{\|\nabla \cdot \mathbf{b}\|^2}{\mu_0}\|\varepsilon_h\|^2 + \frac{\mu_0}{4}\|e_h\|^2$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla \varphi) + (\mathbf{b} \cdot \nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla \varphi_h) + (\mathbf{b} \cdot \nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$(\partial_t e_h, e_h) + \nu(\nabla e_h, \nabla e_h) \qquad\qquad + \mu_0 \|e_h\|^2 =$$

$$(\partial_t \varepsilon_h, e_h) + (c\varepsilon_h, e_h) + (\mathbf{b} \cdot \nabla \varepsilon_h, e_h),$$

$$(\mathbf{b} \cdot \nabla \varepsilon_h, e_h) = -(\varepsilon_h, \mathbf{b} \cdot \nabla e_h) - ((\nabla \cdot \mathbf{b})\varepsilon_h, e_h)$$

$$\leq \frac{\nu}{2}\|\nabla e_h\|^2 + \frac{\|\mathbf{b}\|_\infty^2}{2\nu}\|\varepsilon_h\|^2 + \frac{\|\nabla \cdot \mathbf{b}\|^2}{\mu_0}\|\varepsilon_h\|^2 + \frac{\mu_0}{4}\|e_h\|^2$$

Analysis: Convection-diffusion eqn.

$$(u_t, \varphi) + \nu(\nabla u, \nabla\varphi) + (\mathbf{b}\cdot\nabla u, \varphi) + (cu, \varphi) = (f, \varphi), \qquad \forall\varphi \in H_0^1(\Omega).$$

$$(\partial_t u_h, \varphi_h) + \nu(\nabla u_h, \nabla\varphi_h) + (\mathbf{b}\cdot\nabla u_h, \varphi_h) + (cu_h, \varphi_h) = (f, \varphi_h), \quad \forall\varphi_h \in V_h,$$

for $e_h = \pi_h u - u_h$

$$\|e_h(t)\|^2 \le e^{-\mu_0 t}\|e_h(0)\|^2 + \underbrace{\frac{4}{\mu_0}\int_0^t \left(\|\partial_t\varepsilon_h\|^2 + \mu_1^2\|\varepsilon_h\|^2\right)\,dt}_{O(h^{2(k+1)})} + \underbrace{\frac{2}{\mu_0}\int_0^t \|\mathbf{b}\cdot\nabla\varepsilon_h\|^2\,dt}_{O(h^{2k})}$$

$$\|e_h(t)\|^2 \le e^{-\mu_0 t}\|e_h(0)\|^2 + \underbrace{\frac{8}{\mu_0}\int_0^t \left(\|\partial_t\varepsilon_h\|^2 + C^2\|\varepsilon_h\|^2\right)\,dt}_{O(h^{2(k+1)})} + \underbrace{\frac{1}{\nu}\int_0^t \|\mathbf{b}\|_\infty^2\|\varepsilon_h\|^2\,dt}_{O(\nu^{-1}h^{2(k+1)})}$$

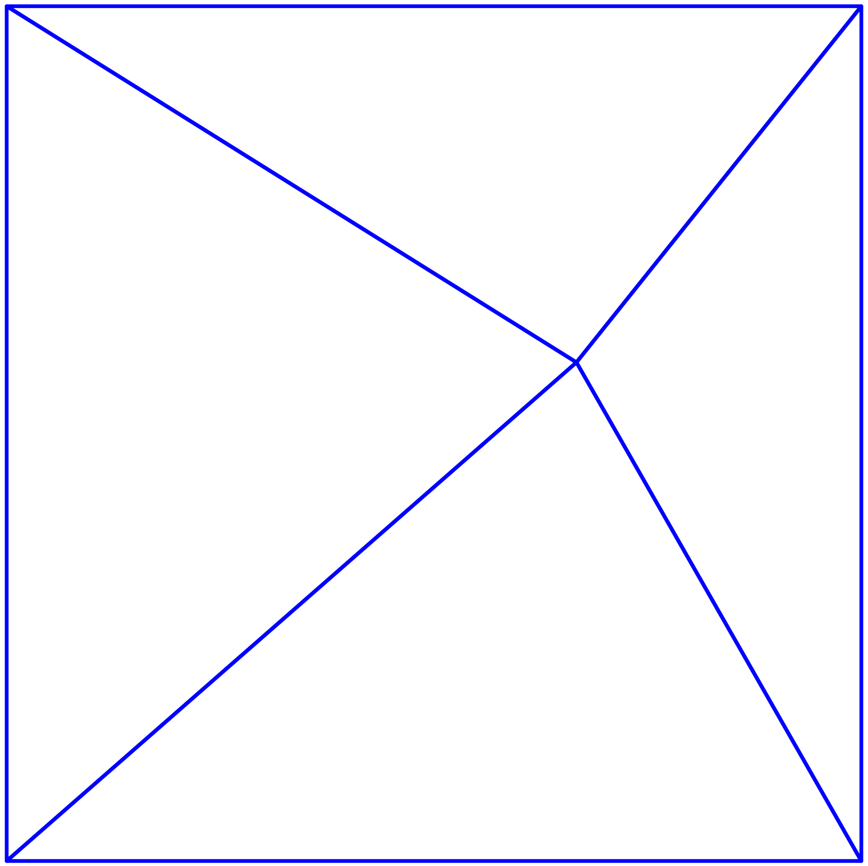
Convection-diffusion eqn. Example

$$
\begin{aligned}
u_t - \nu\Delta u + \mathbf{b}\cdot\nabla u + cu &= f && \text{in } (0,T]\times\Omega,\\
u(0,\cdot) &= u_0(\cdot) && \text{in } \Omega,\\
u &= 0 && \text{on } [0,T]\times\partial\Omega,
\end{aligned}
$$

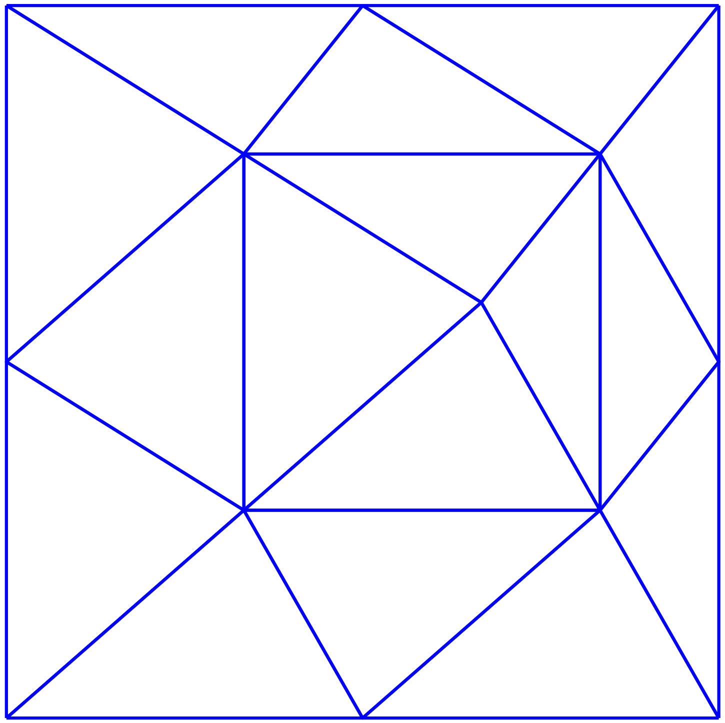
$$
\begin{aligned}
\Omega = (0,1)\times(0,1), \qquad & T = 2,\\
\mathbf{b} = (1-y, -2/3+x)^T, \quad & c = 1,
\end{aligned}
$$

f chosen so that

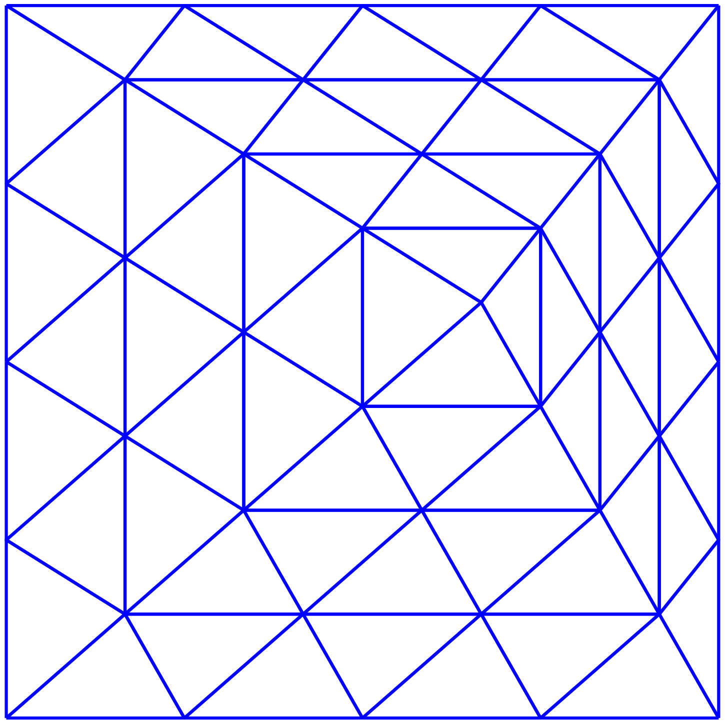
$$u(x,t) = \sin(\pi t)\sin(2\pi x)\sin(2\pi y).$$



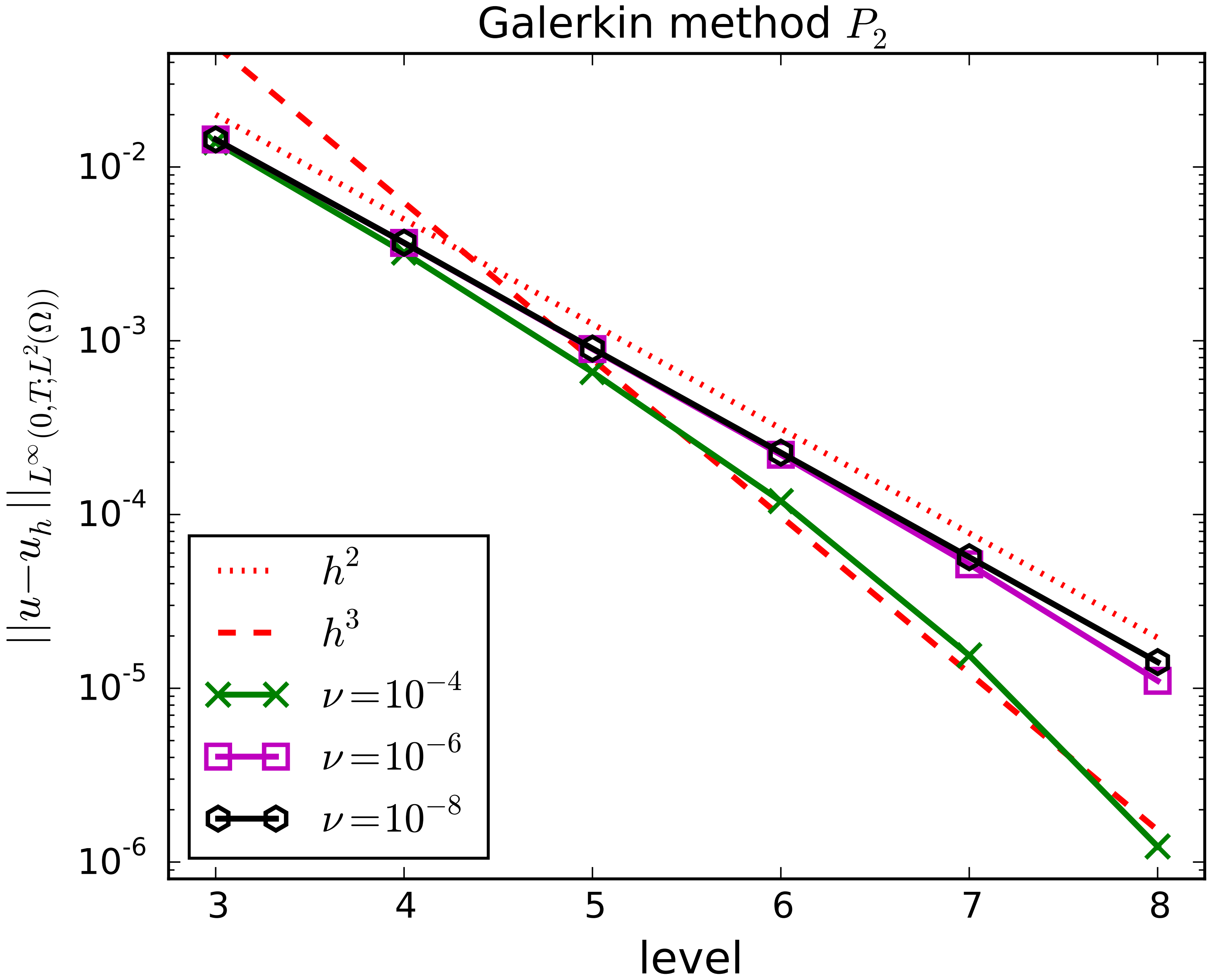
Initial grid (level 1).



First refined grid (level 2)



Second refined grid (level 3)



Galerkin method P_2
$||u-u_h||_{L^\infty(0,T;L^2(\Omega))}$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
3
4
5
6
7
8
level
h^2
h^3
$\nu=10^{-4}$
$\nu=10^{-6}$
$\nu=10^{-8}$

Convection-diffusion eqn. Stabilized methods

SUPG: Streamline Upwind Petrov Galerkin method $O(h^{k+1/2})$ error est.

Steady problems

- Brooks & Hughes (1982), Hughes & Brooks 1979
- Roos, Stynes & Tobiska (2008).

Time-dependent problems

- Hughes, Franca & Mallet (1987) (Space-time elem., $\Delta t > ch$)
- N-S eqs.: Hansbo & Szpessy (1990), Lube & Tobiska (1990) (Idem)
- Burman (2010) ($\Delta t > ch$)
- John & Novo (2011)

Analysis: Convection-diffusion eqn. Stabilized methods (term by term LPS

$$(u_t,\varphi)+\nu(\nabla u,\nabla\varphi)+(\mathbf{b}\cdot\nabla u+cu,\varphi)=(f,\varphi),\qquad \forall\varphi\in H_0^1(\Omega).$$

$$(\partial_t u_h,\varphi_h)+\nu(\nabla u_h,\nabla\varphi_h)+(\mathbf{b}\cdot\nabla u_h+cu_h,\varphi_h)+S_h(u_h,\varphi_h)=(f,\varphi_h),\quad \forall\varphi_h\in V_h,$$

(Details after the talk if requested)

Analysis: Convection-diffusion eqn. Stabilized methods

$$(u_t, \varphi) + \nu(\nabla u, \nabla\varphi) + (\mathbf{b}\cdot\nabla u + cu, \varphi) = (f, \varphi), \qquad \forall\varphi \in H_0^1(\Omega).$$

Term by term stabilization (variant of LPS stabilization)

Chacón Rebollo, Gómez Marmol, Girault & Sánchez Muñoz (2013)

Ahmed, Chacón Rebollo, V. John & Rubino (2017). N-S eqns.

(Error bounds depending on ν^{-1})

de Frutos, G-A, John & Novo (2019). N-S eqns.

(Error bounds independent of ν)

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \qquad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,\\
(\nabla\cdot\mathbf{u}, \psi) &= 0, \qquad\qquad \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) =&(\mathbf{f}, \boldsymbol{\varphi}_h),\\
&\boldsymbol{\varphi}_h \in \mathbf{V}_h,\\
(\nabla\cdot\mathbf{u}_h, \psi_h) =&0, \qquad \psi_h \in Q_h,
\end{aligned}
$$

where $\qquad B(\mathbf{u}, \mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in H_0^1(\Omega)^d$

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad && \boldsymbol{\varphi} \in H_0^1(\Omega)^d, \\
(\nabla\cdot\mathbf{u}, \psi) &= 0, && \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) =& (\mathbf{f}, \boldsymbol{\varphi}_h), \\
& \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
(\nabla\cdot\mathbf{u}_h, \psi_h) =& 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

Difficulties:

The Navier-Stokes equations: Analysis

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) = (\boldsymbol{f}, \mathbf{v}), \qquad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,$$

$$(\nabla\cdot\mathbf{u}, \psi) = 0, \qquad \psi \in L_0^2(\Omega),$$

$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h),$$

$$\boldsymbol{\varphi}_h \in \mathbf{V}_h,$$

$$(\nabla\cdot\mathbf{u}_h, \psi_h) = 0, \qquad \psi_h \in Q_h,$$

Difficulties:

1 - Pressure and incompressibility condition.

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad && \boldsymbol{\varphi} \in H_0^1(\Omega)^d, \\
(\nabla\cdot\mathbf{u}, \psi) &= 0, && \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) =& (\mathbf{f}, \boldsymbol{\varphi}_h), \\
& \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
(\nabla\cdot\mathbf{u}_h, \psi_h) =& 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

Difficulties:

1 - Pressure and incompressibility condition.

2 - Convection and diffusion when $\nu \ll 1$ ($\mathrm{Re} \gg 1$).

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad && \boldsymbol{\varphi} \in H_0^1(\Omega)^d, \\
(\nabla\cdot\mathbf{u}, \psi) &= 0, && \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) &= (\mathbf{f}, \boldsymbol{\varphi}_h), \\
& \qquad \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
(\nabla\cdot\mathbf{u}_h, \psi_h) &= 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

Difficulties:

1 - Pressure and incompressibility condition.

2 - Convection and diffusion when $\nu \ll 1$ ($\mathrm{Re} \gg 1$).

3 - Nonlinearity.

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad && \boldsymbol{\varphi} \in H_0^1(\Omega)^d, \\
(\nabla\cdot\mathbf{u}, \psi) &= 0, && \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) =& (\mathbf{f}, \boldsymbol{\varphi}_h), \\
& \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
(\nabla\cdot\mathbf{u}_h, \psi_h) =& 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

Pressure and incompressibility condition.

The Navier-Stokes equations: Analysis

$$\begin{array}{rcll} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &=& \mathbf{f}, & \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &=& 0, & \text{in } (0,T]\times\Omega, \\ \mathbf{u}(0,\cdot) &=& \mathbf{u}_0(\cdot) & \text{in } \Omega, \\ \mathbf{u} &=& 0, & \text{on } (0,T]\times\partial\Omega. \end{array}$$

$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla\mathbf{u}_h, \nabla\boldsymbol{\varphi}_h) + (B(\mathbf{u}_h,\mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h), \quad \boldsymbol{\varphi}_h \in \mathbf{V}_h,$$

$$(\nabla\cdot\mathbf{u}_h, \psi) = 0, \quad \psi \in L^2(\Omega),$$

Pressure and incompressibility condition. Not a difficulty if div-free elements

Analysis (after talk) and results similar to convection-diffusion problems.

The Navier-Stokes equations: Example

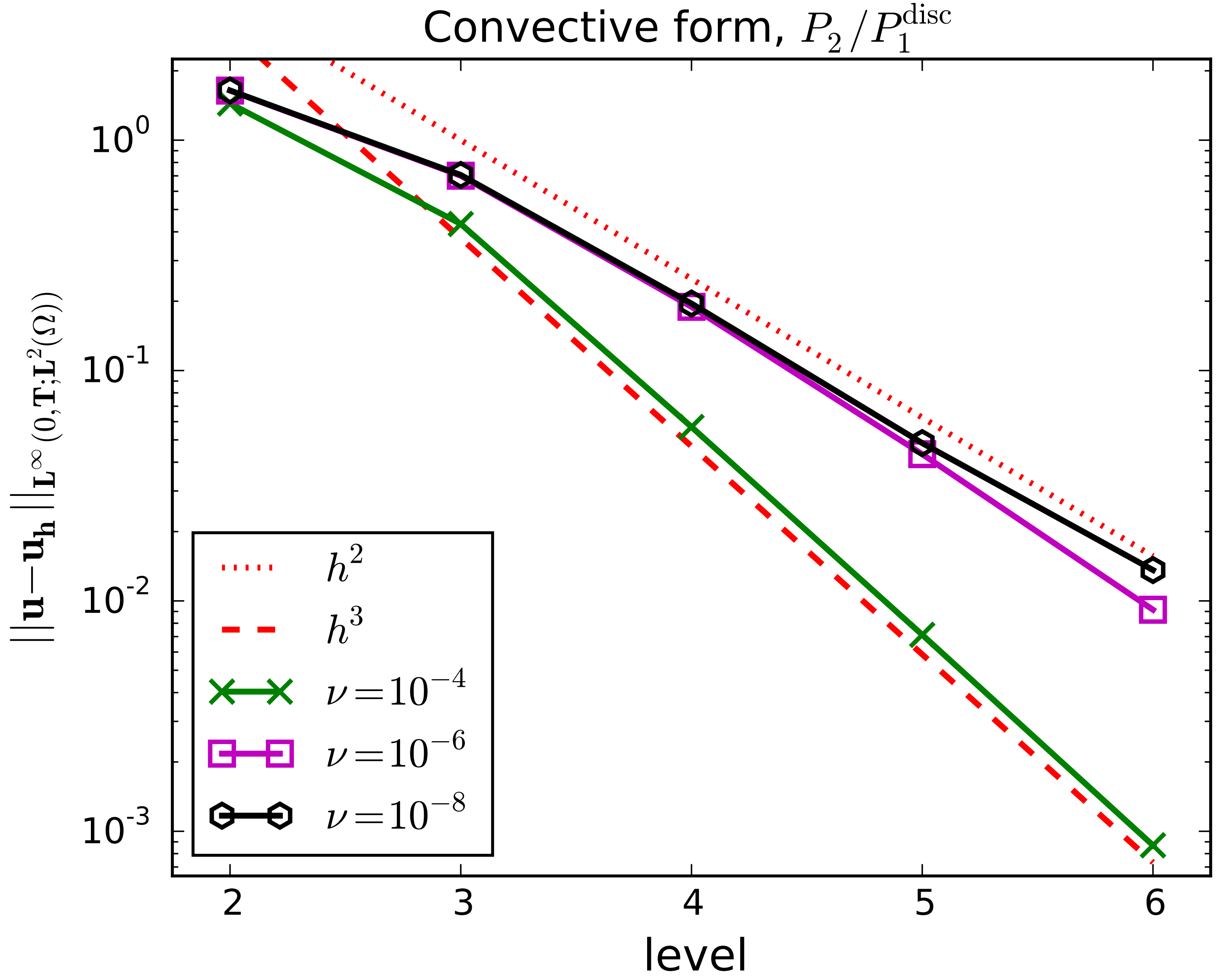
$$
\begin{array}{rcll}
u_t - \nu \Delta u + \mathbf{b} \cdot \nabla u + cu & = & f & \text{in } (0,T] \times \Omega, \\
u(0,\cdot) & = & u_0(\cdot) & \text{in } \Omega, \\
u & = & 0 & \text{on } [0,T] \times \partial\Omega,
\end{array}
$$

$$
\Omega = (0,1) \times (0,1), \qquad T = 2,
$$

f chosen so that

$$
\mathbf{u}(\mathbf{x},t) = 2\pi \sin(\pi t) \begin{bmatrix} \sin^2(\pi x)\sin(\pi y)\cos(\pi y) \\ -\sin^2(\pi y)\sin(\pi x)\cos(\pi x) \end{bmatrix},
$$

$$
p(\mathbf{x},t) = \frac{10}{3}\sin(\pi t)(6x^2 y - 1).
$$



Convective form, P_2/P_1^{disc}
$||\mathbf{u}-\mathbf{u_h}||_{\mathbf{L}^\infty(0,\mathbf{T};\mathbf{L}^2(\Omega))}$
10^0
10^{-1}
10^{-2}
10^{-3}
2
3
4
5
6
level
h^2
h^3
$\nu=10^{-4}$
$\nu=10^{-6}$
$\nu=10^{-8}$

Degrees of freedom

Divergence-free Scott-Vogelius P2/P1$^{\mathrm{disc}}$ vs Taylor-Hood P2/P1.

magnitude/ level	4	5	6	7	8
velocity	3010	12162	48898	196098	785410
pressure (div-free)	2304	9216	36864	147456	589824
pressure (T-H)	401	1569	6209	24705	98561

A. Linke & C. Merdon, Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier-Stokes equations. CMAME, 311 (2016), 304–326.

"Computationally very expensive method, especially in 3D"

"Only in problems with very difficult pressures is competitive with other finite element methods like Taylor-Hood"

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad && \boldsymbol{\varphi} \in H_0^1(\Omega)^d, \\
(\nabla \cdot \mathbf{u}, \psi) &= 0, && \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + (B(\mathbf{u}_h, \mathbf{u}_h), \boldsymbol{\varphi}_h) - (p_h, \nabla \cdot \boldsymbol{\varphi}_h) =& (\mathbf{f}, \boldsymbol{\varphi}_h), \\
& \qquad \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
(\nabla \cdot \mathbf{u}_h, \psi_h) =& 0, \qquad \psi_h \in Q_h,
\end{aligned}
$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements.

The Navier-Stokes equations: Analysis

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) \qquad - (p, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,$$
$$(\nabla \cdot \mathbf{u}, \psi) = 0, \qquad \psi \in L_0^2(\Omega),$$

$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) \qquad - (p_h, \nabla \cdot \boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h),$$
$$\boldsymbol{\varphi}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \psi_h) = 0, \quad \psi_h \in Q_h,$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements.

The Navier-Stokes equations: Analysis

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,$$
$$(\nabla \cdot \mathbf{u}, \psi) = 0, \qquad \psi \in L_0^2(\Omega),$$

$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) - (p_h, \nabla \cdot \boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h),$$
$$\boldsymbol{\varphi}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{u}_h, \psi_h) = 0, \quad \psi_h \in Q_h,$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements. For $\mathbf{e}_h = \mathbf{u}_h - \boldsymbol{s}_h$, $\boldsymbol{\varepsilon}_h = \mathbf{u} - \boldsymbol{s}_h$ and $\pi_h p$ best approx.,

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{e}_h\|^2 + \nu\|\nabla \mathbf{e}_h\|^2 \leq (\partial_t \boldsymbol{\varepsilon}_h, \mathbf{e}_h) + (p - \pi_h p, \nabla \cdot \mathbf{e}_h),$$

$$(p - \pi_h p, \nabla \cdot \mathbf{e}_h) \leq \frac{\|p - \pi_h p\|^2}{2\nu} + \frac{\nu}{2}\|\nabla \mathbf{e}_h\|^2,$$

The Navier-Stokes equations: Analysis

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) \qquad - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,\\
(\nabla\cdot\mathbf{u}, \psi) &= 0, \qquad\quad \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) \qquad - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) =& (\mathbf{f}, \boldsymbol{\varphi}_h),\\
& \boldsymbol{\varphi}_h \in \mathbf{V}_h,\\
(\nabla\cdot\mathbf{u}_h, \psi_h) =& 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements. For $\mathbf{e}_h = \mathbf{u}_h - \boldsymbol{s}_h$, $\boldsymbol{\varepsilon}_h = \mathbf{u} - \boldsymbol{s}_h$ and $\pi_h p$ best approx., $L = O(1)$,

$$
\frac{d}{dt}\frac{1}{2}\|\mathbf{e}_h\|^2 + \nu\|\nabla \mathbf{e}_h\|^2 \le (\partial_t \boldsymbol{\varepsilon}_h, \mathbf{e}_h) + (p - \pi_h p, \nabla\cdot\mathbf{e}_h),
$$

$$
(p - \pi_h p, \nabla\cdot\mathbf{e}_h) = -(\nabla(p - \pi_h p, \mathbf{e}_h) \le \frac{1}{L}\|\nabla(p - \pi_h p)\|^2 + \frac{L}{4}\|\mathbf{e}_h\|^2,
$$

The Navier-Stokes equations: Analysis

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) - (p, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,$$

$$(\nabla \cdot \mathbf{u}, \psi) = 0, \qquad \psi \in L_0^2(\Omega),$$

$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) - (p_h, \nabla \cdot \boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h), \quad \boldsymbol{\varphi}_h \in \mathbf{V}_h,$$

$$(\nabla \cdot \mathbf{u}_h, \psi_h) = 0, \quad \psi_h \in Q_h,$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements. For $\mathbf{e}_h = \mathbf{u}_h - \boldsymbol{s}_h$, $\boldsymbol{\varepsilon}_h = \mathbf{u} - \boldsymbol{s}_h$ and $\pi_h p$ best approx., $L = O(1)$,

$$\|\mathbf{e}_h(t)\|^2 \leq e^{Lt}\Bigg(\|\mathbf{e}_h(0)\|^2 + \underbrace{\frac{1}{L}\int_0^t \|\partial_t \boldsymbol{\varepsilon}_h\|^2\, dt}_{O(h^{2(k+1)})} + \underbrace{\frac{1}{\nu}\int_0^t \|p - \pi_h p\|^2\, dt.}_{O(\nu^{-1} h^{2k})}\Bigg)$$

The Navier-Stokes equations: Analysis

$$(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) - (p, \nabla\cdot\boldsymbol{\varphi}) = (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,$$

$$(\nabla\cdot\mathbf{u}, \psi) = 0, \quad \psi \in L_0^2(\Omega),$$

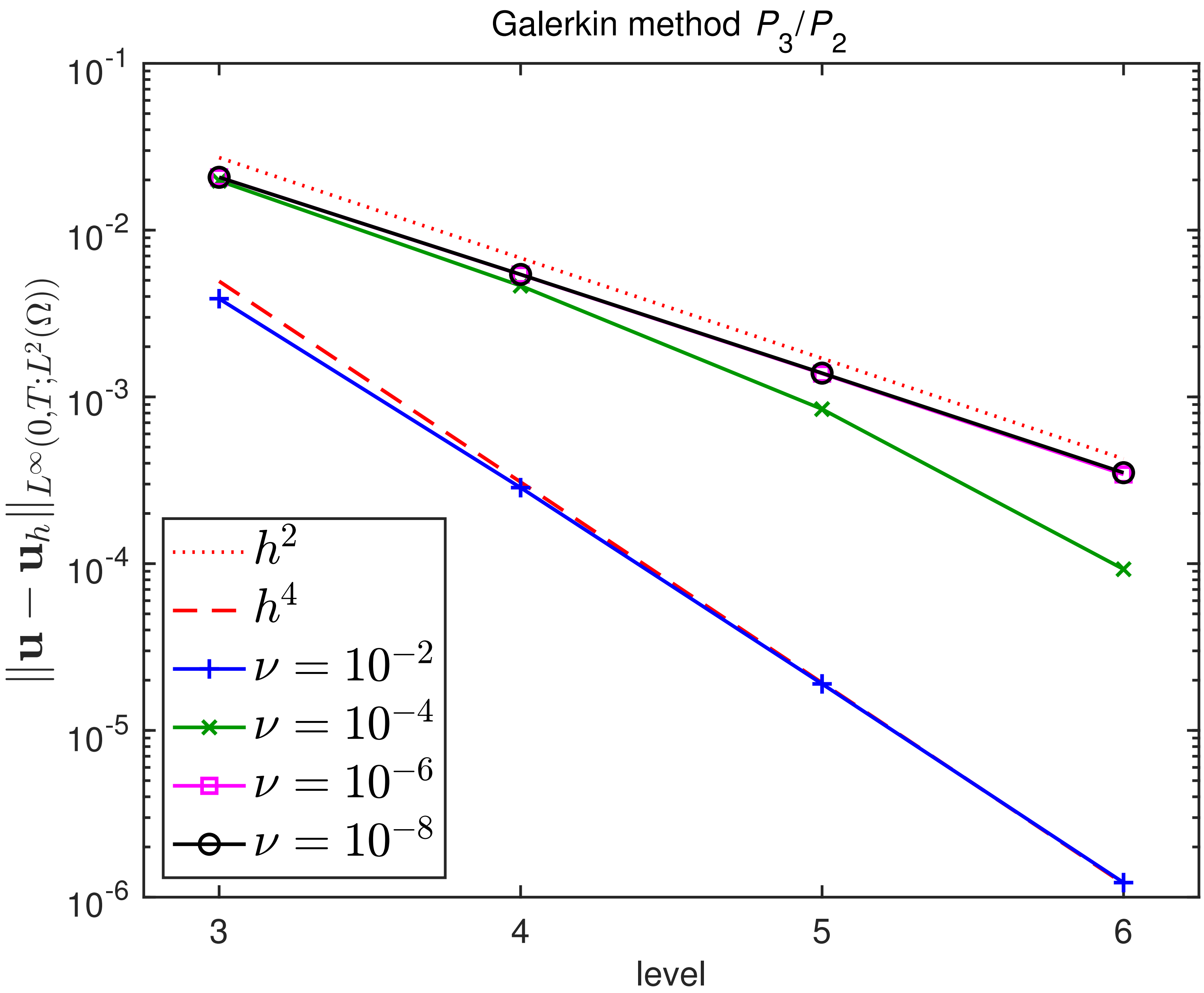
$$((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) = (\mathbf{f}, \boldsymbol{\varphi}_h),$$

$$\boldsymbol{\varphi}_h \in \mathbf{V}_h,$$

$$(\nabla\cdot\mathbf{u}_h, \psi_h) = 0, \quad \psi_h \in Q_h,$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements. For $\mathbf{e}_h = \mathbf{u}_h - \boldsymbol{s}_h$, $\boldsymbol{\varepsilon}_h = \mathbf{u} - \boldsymbol{s}_h$ and $\pi_h p$ best approx., $L = O(1)$,

$$\|\mathbf{e}_h(t)\|^2 \le e^{Lt}\left(\|\mathbf{e}_h(0)\|^2 + \underbrace{\frac{2}{L}\int_0^t \|\partial_t \boldsymbol{\varepsilon}_h\|^2\, dt}_{O(h^{2(k+1)})} + \underbrace{\frac{2}{L}\int_0^t \|\nabla(p - \pi_h p)\|^2\, dt.}_{O(h^{2(k-1)})}\right)$$



Galerkin method P_3/P_2
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$
3
4
5
6
level
h^2
h^4
$\nu = 10^{-2}$
$\nu = 10^{-4}$
$\nu = 10^{-6}$
$\nu = 10^{-8}$

The Navier-Stokes equations: Analysis with grad-div stabilization

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) \qquad\qquad - (p, \nabla \cdot \boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d, \\
(\nabla \cdot \mathbf{u}, \psi) &= 0, \qquad\quad \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + \mu(\nabla \cdot \mathbf{u}_h, \nabla \cdot \boldsymbol{\varphi}_h) - (p_h, \nabla \cdot \boldsymbol{\varphi}_h) =& (\mathbf{f}, \boldsymbol{\varphi}_h), \\
& \boldsymbol{\varphi}_h \in \mathbf{V}_h, \\
(\nabla \cdot \mathbf{u}_h, \psi_h) =& 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements. For $\mathbf{e}_h = \mathbf{u}_h - \boldsymbol{s}_h$, $\boldsymbol{\varepsilon}_h = \mathbf{u} - \boldsymbol{s}_h$ and $\pi_h p$ best approx.,

$$
\frac{d}{dt}\frac{1}{2}\|\mathbf{e}_h\|^2 + \nu\|\nabla \mathbf{e}_h\|^2 + \mu\|\nabla \cdot \mathbf{e}_h\|^2 \leq (\partial_t \boldsymbol{\varepsilon}_h, \mathbf{e}_h) + (p - \pi_h p, \nabla \cdot \mathbf{e}_h),
$$

$$
(p - \pi_h p, \nabla \cdot \mathbf{e}_h) \leq \frac{\|p - \pi_h p\|^2}{2\mu} + \frac{\mu}{2}\|\nabla \cdot \mathbf{e}_h\|^2,
$$

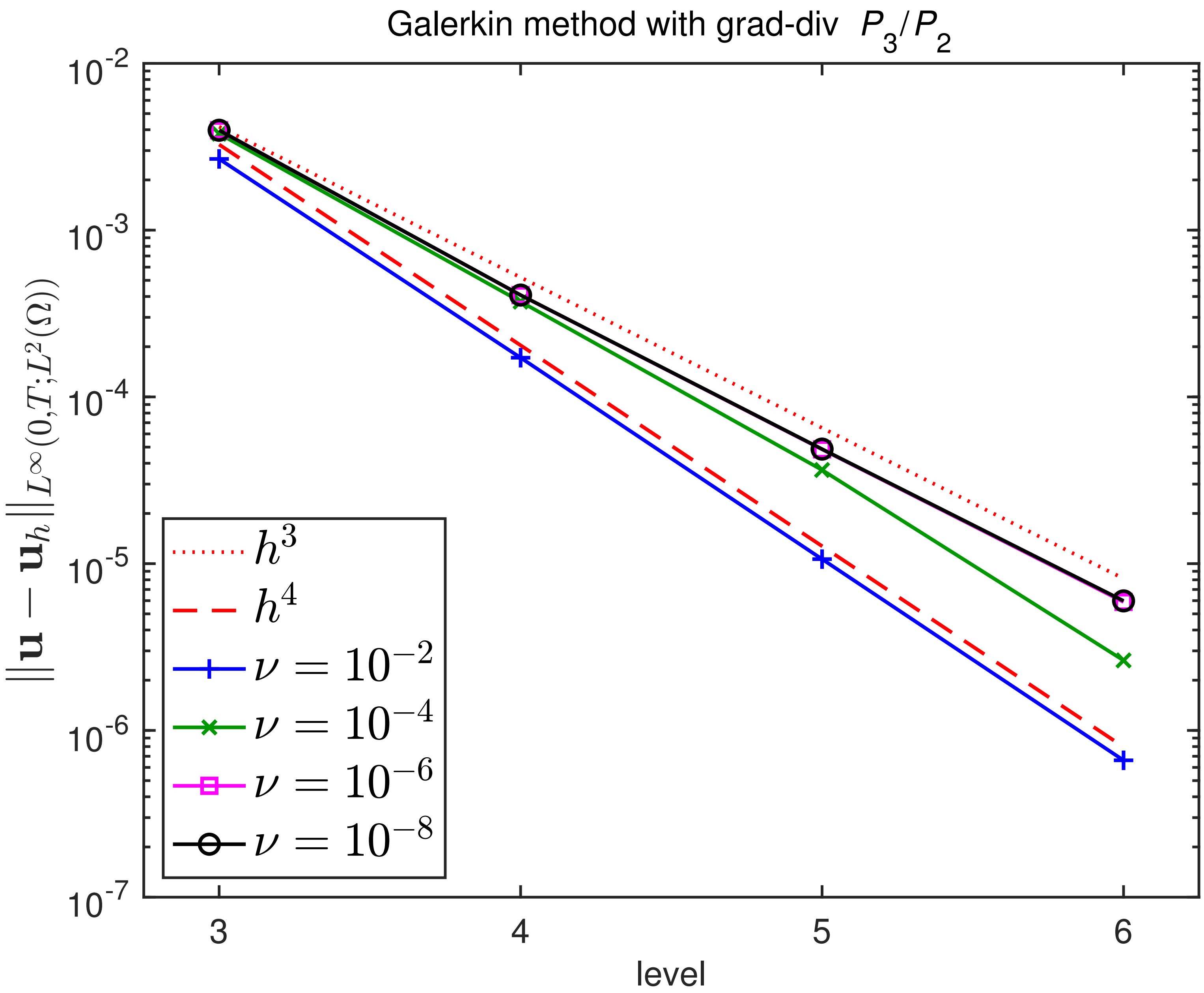
The Navier-Stokes equations: Analysis with grad-div stabilization

$$
\begin{aligned}
(\partial_t \mathbf{u}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}, \nabla \boldsymbol{\varphi}) \qquad - (p, \nabla\cdot\boldsymbol{\varphi}) &= (\boldsymbol{f}, \mathbf{v}), \quad \boldsymbol{\varphi} \in H_0^1(\Omega)^d,\\
(\nabla\cdot\mathbf{u}, \psi) &= 0, \qquad\quad \psi \in L_0^2(\Omega),
\end{aligned}
$$

$$
\begin{aligned}
((\mathbf{u}_h)_t, \boldsymbol{\varphi}_h) + \nu(\nabla \mathbf{u}_h, \nabla \boldsymbol{\varphi}_h) + \mu(\nabla\cdot\mathbf{u}_h, \nabla\cdot\boldsymbol{\varphi}_h) - (p_h, \nabla\cdot\boldsymbol{\varphi}_h) &= (\mathbf{f}, \boldsymbol{\varphi}_h),\\
&\qquad \boldsymbol{\varphi}_h \in \mathbf{V}_h,\\
(\nabla\cdot\mathbf{u}_h, \psi_h) &= 0, \quad \psi_h \in Q_h,
\end{aligned}
$$

1 - Pressure and incompressibility condition. Case of non weakly div-free elements. For $\mathbf{e}_h = \mathbf{u}_h - \boldsymbol{s}_h$, $\boldsymbol{\varepsilon}_h = \mathbf{u} - \boldsymbol{s}_h$ and $\pi_h p$ best approx., $L = O(1)$,

$$
\|\mathbf{e}_h(t)\|^2 \le e^{Lt}\left(\|\mathbf{e}_h(0)\|^2 + \underbrace{\frac{1}{L}\int_0^t \|\partial_t \boldsymbol{\varepsilon}_h\|^2\, dt}_{O(h^{2(k+1)})} + \underbrace{\frac{1}{\mu}\int_0^t \|(p - \pi_h p)\|^2\, dt.}_{O(h^{2k})} \right)
$$



Galerkin method with grad-div P_3/P_2
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$
3
4
5
6
level
h^3
h^4
$\nu = 10^{-2}$
$\nu = 10^{-4}$
$\nu = 10^{-6}$
$\nu = 10^{-8}$

The Navier-Stokes equations: Analysis

3 - The effect of the nonlinearity. Adding $\pm(B(\mathbf{u}_h, \boldsymbol{s}_h), \mathbf{e}_h)$,

$$
\begin{aligned}
(B(\mathbf{u}_h, \mathbf{u}_h) - B(\boldsymbol{s}_h, \boldsymbol{s}_h), \mathbf{e}_h) &= \underbrace{B(\mathbf{u}_h, \mathbf{e}_h, \mathbf{e}_h)}_{=0} + B(\mathbf{e}_h, \boldsymbol{s}_h, \mathbf{e}_h) \\
&= ((\mathbf{e}_h \cdot \nabla)\boldsymbol{s}_h, \mathbf{e}_h) + \frac{1}{2}(\nabla \cdot \mathbf{e}_h, \boldsymbol{s}_h \cdot \mathbf{e}_h) \\
&\leq \left(\|\nabla \boldsymbol{s}_h\|_\infty + \frac{\|\boldsymbol{s}_h\|_\infty^2}{4\nu}\right)\|\mathbf{e}_h\|^2 + \frac{\nu}{4}\|\nabla \mathbf{e}_h\|^2.
\end{aligned}
$$

The Navier-Stokes equations: Analysis

3 - The effect of the nonlinearity

$$
\begin{aligned}
(B(\mathbf{u}_h,\mathbf{u}_h) - B(\boldsymbol{s}_h,\boldsymbol{s}_h),\mathbf{e}_h) &= \underbrace{B(\mathbf{u}_h,\mathbf{e}_h,\mathbf{e}_h)}_{=0} + B(\mathbf{e}_h,\boldsymbol{s}_h,\mathbf{e}_h) \\
&= ((\mathbf{e}_h\cdot\nabla)\boldsymbol{s}_h,\mathbf{e}_h) + \frac{1}{2}(\nabla\cdot\mathbf{e}_h,\boldsymbol{s}_h\cdot\mathbf{e}_h) \\
&\le \underbrace{\frac{1}{\nu}\left(\nu\|\nabla\boldsymbol{s}_h\|_\infty + \frac{\|\boldsymbol{s}_h\|_\infty^2}{4}\right)}_{L'}\|\mathbf{e}_h\|^2 + \frac{\nu}{4}\|\nabla\mathbf{e}_h\|^2.
\end{aligned}
$$

The Navier-Stokes equations: Analysis

3 - The effect of the nonlinearity

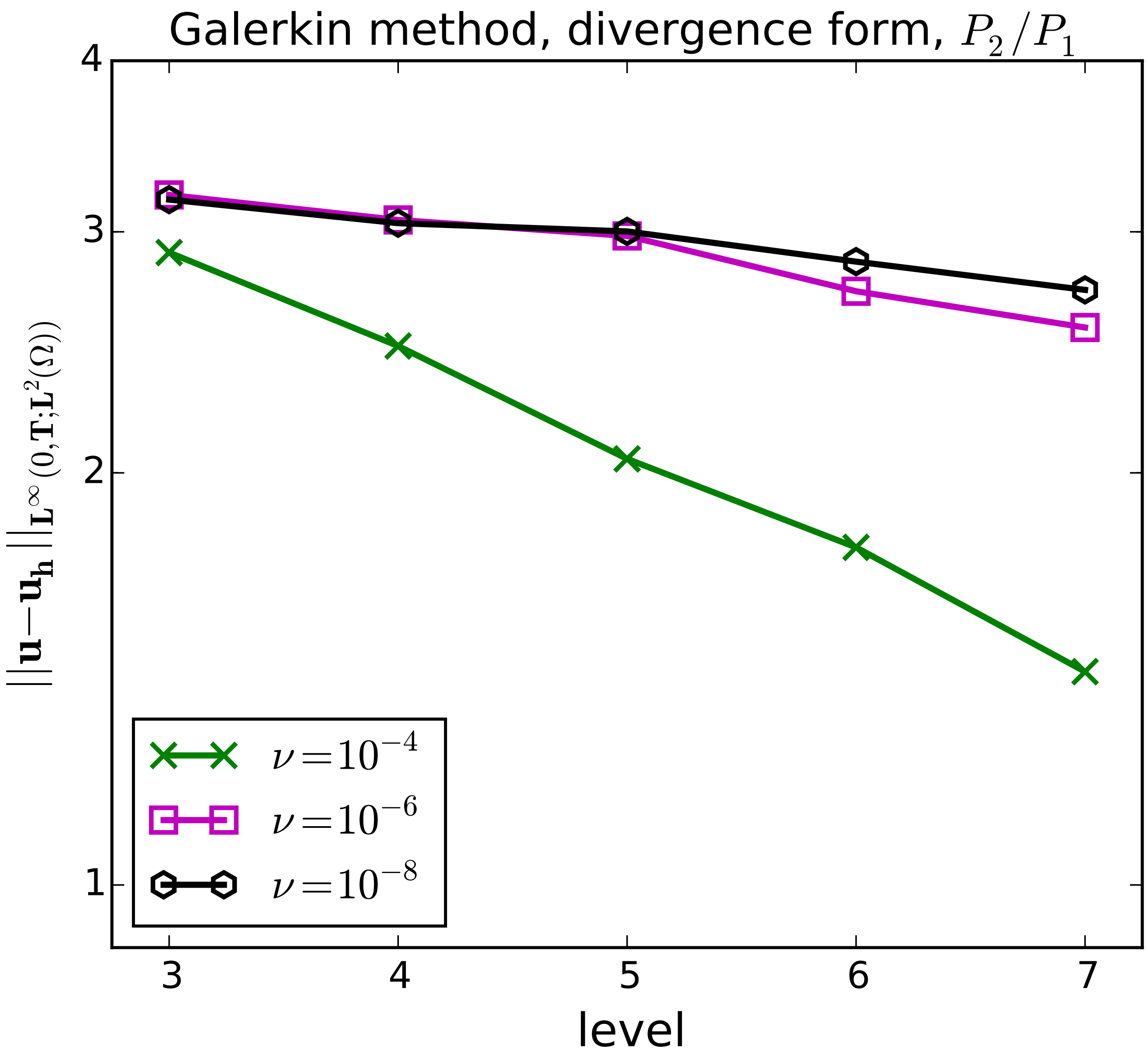
$$(B(\mathbf{u}_h, \mathbf{u}_h) - B(\boldsymbol{s}_h, \boldsymbol{s}_h), \mathbf{e}_h) = \underbrace{B(\mathbf{u}_h, \mathbf{e}_h, \mathbf{e}_h)}_{=0} + B(\mathbf{e}_h, \boldsymbol{s}_h, \mathbf{e}_h)$$

$$= ((\mathbf{e}_h \cdot \nabla)\boldsymbol{s}_h, \mathbf{e}_h) + \frac{1}{2}(\nabla \cdot \mathbf{e}_h, \boldsymbol{s}_h \cdot \mathbf{e}_h)$$

$$\leq \frac{1}{\nu}\left(\nu \|\nabla \boldsymbol{s}_h\|_\infty + \frac{\|\boldsymbol{s}_h\|_\infty^2}{4}\right)\|\mathbf{e}_h\|^2 + \frac{\nu}{4}\|\nabla \mathbf{e}_h\|^2.$$

$$\|\mathbf{e}_h(t)\|^2 \leq e^{(L+(L'/\nu))t}\Bigg(\|\mathbf{e}_h(0)\|^2 + \underbrace{\frac{1}{L}\int_0^t \|\partial_t \varepsilon_h\|^2\,dt}_{O(h^{2(k+1)})} + \underbrace{\frac{1}{L}\int_0^t \|\nabla(p - \pi_h p)\|^2\,dt.}_{O(h^{2(k-1)})}$$

$$+ \underbrace{\frac{C^2}{L}\int_0^t \|\nabla \varepsilon_h\|^2\,dt}_{O(h^{2k})}\Bigg).$$

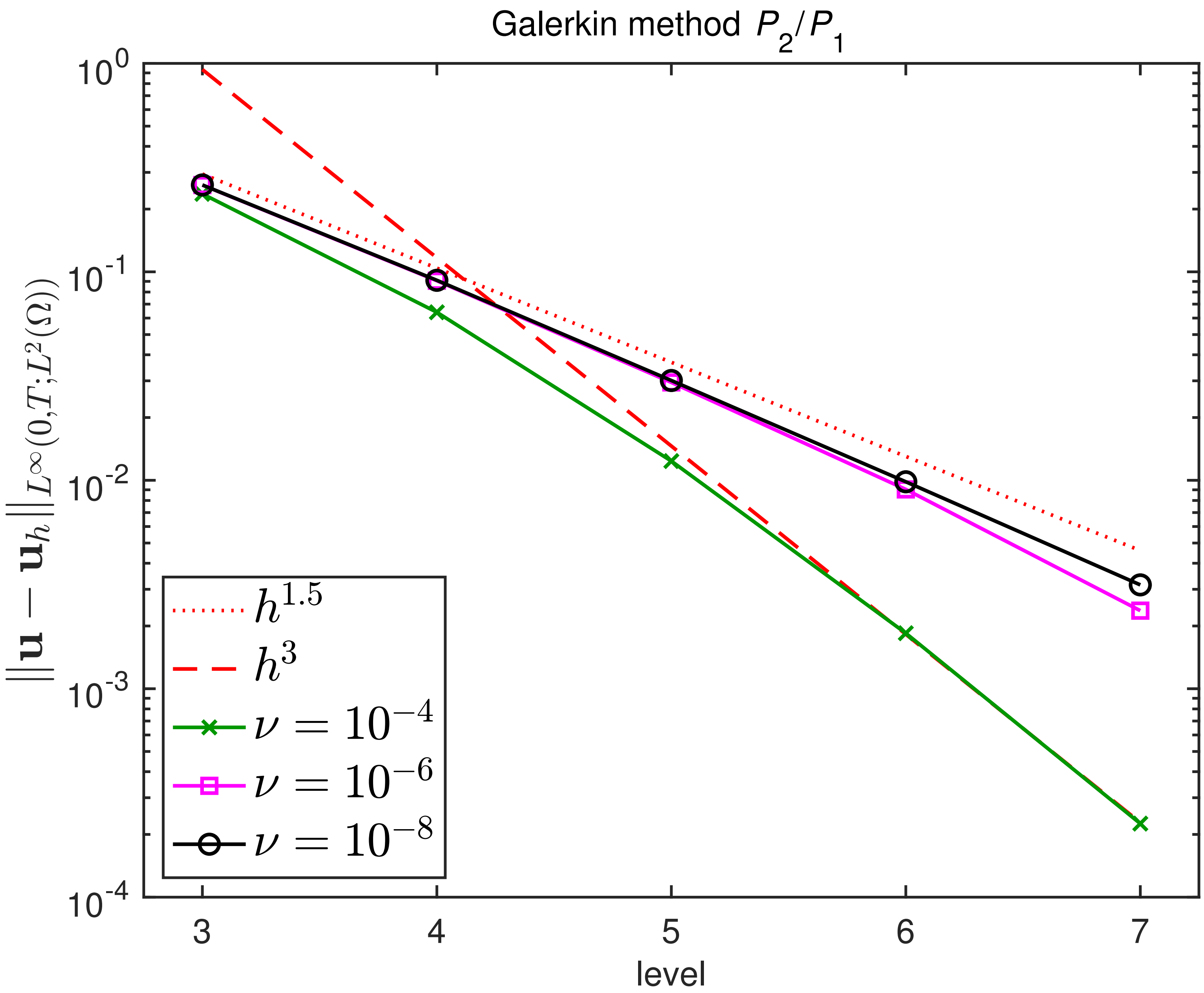


Galerkin method, divergence form, P_2/P_1
$||\mathbf{u}-\mathbf{u_h}||_{\mathbf{L}^\infty(0,\mathbf{T};\mathbf{L}^2(\Omega))}$
level
4
3
2
1
3
4
5
6
7
$\nu = 10^{-4}$
$\nu = 10^{-6}$
$\nu = 10^{-8}$

The Oseen equations

$$\begin{aligned}
\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q &= \mathbf{f}, \quad \text{in } (0,T]\times\Omega, \\
\nabla\cdot\mathbf{v} &= 0, \quad \text{in } (0,T]\times\Omega, \\
\mathbf{v}(0,\cdot) &= \mathbf{u}_0(\cdot) \quad \text{in } \Omega, \\
\mathbf{v} &= 0, \quad \text{on } (0,T]\times\partial\Omega.
\end{aligned}$$

$$\mathbf{b} = \mathbf{u} \quad \Rightarrow \quad \mathbf{v} = \mathbf{u}.$$



Galerkin method P_2/P_1
$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
3
4
5
6
7
level
$h^{1.5}$
h^3
$\nu=10^{-4}$
$\nu=10^{-6}$
$\nu=10^{-8}$

The Navier-Stokes equations: Analysis with grad-div stabilization

3 - The effect of the nonlinearity

$$
\begin{aligned}
(B(\mathbf{u}_h,\mathbf{u}_h)-B(\boldsymbol{s}_h,\boldsymbol{s}_h),\mathbf{e}_h) &= \underbrace{B(\mathbf{u}_h,\mathbf{e}_h,\mathbf{e}_h)}_{=0} + B(\mathbf{e}_h,\boldsymbol{s}_h,\mathbf{e}_h) \\
&= ((\mathbf{e}_h\cdot\nabla)\boldsymbol{s}_h,\mathbf{e}_h) + \frac{1}{2}(\nabla\cdot\mathbf{e}_h,\boldsymbol{s}_h\cdot\mathbf{e}_h) \\
&\le \underbrace{\left(\|\nabla\boldsymbol{s}_h\|_\infty + \frac{\|\boldsymbol{s}_h\|_\infty^2}{4\mu}\right)}_{L}\|\mathbf{e}_h\|^2 + \frac{\mu}{4}\|\nabla\cdot\mathbf{e}_h\|^2.
\end{aligned}
$$

The Navier-Stokes equations: Analysis with grad-div stabilization

3 - The effect of the nonlinearity

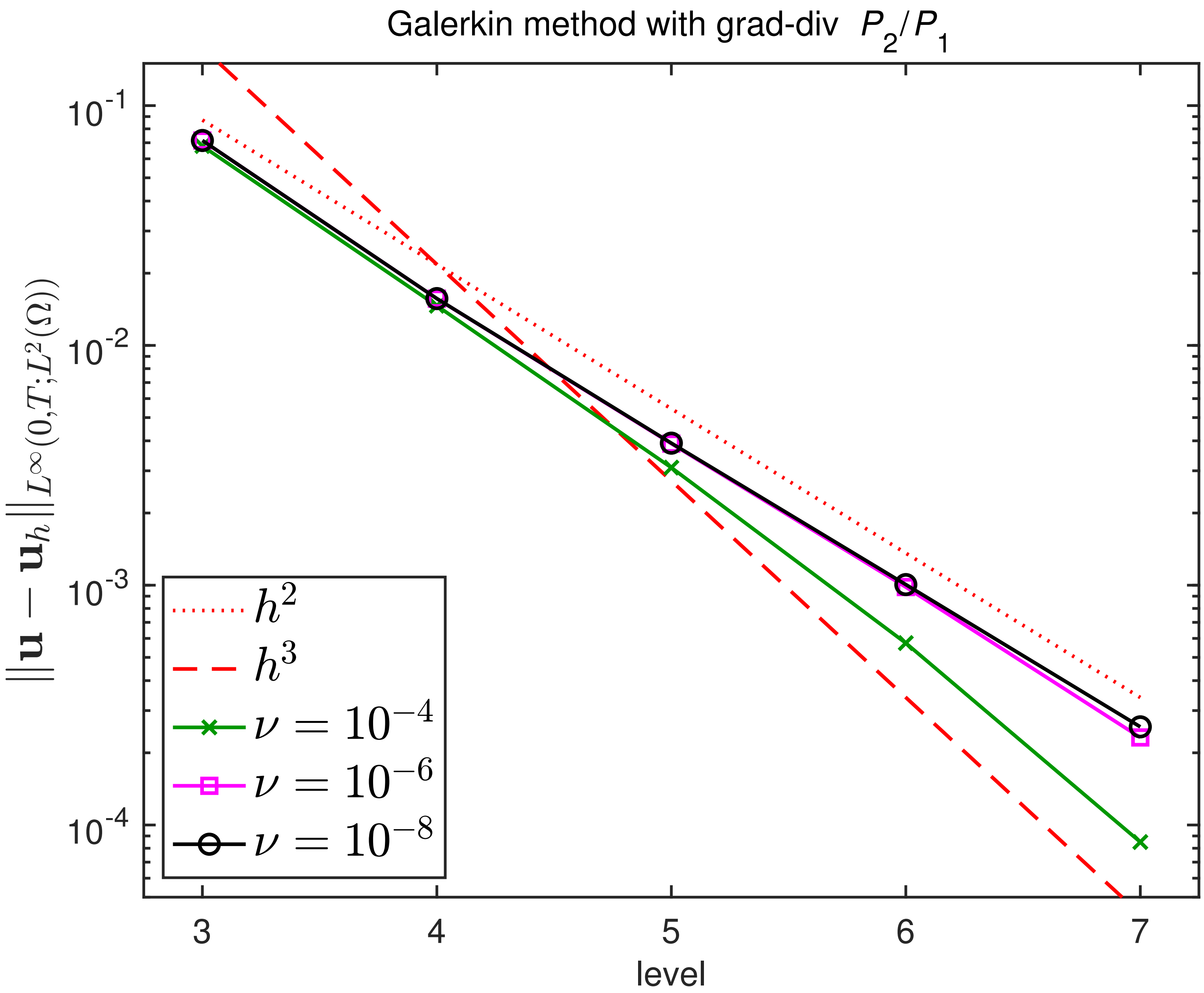
$$(B(\mathbf{u}_h,\mathbf{u}_h) - B(\boldsymbol{s}_h,\boldsymbol{s}_h),\mathbf{e}_h) = \underbrace{B(\mathbf{u}_h,\mathbf{e}_h,\mathbf{e}_h)}_{=0} + B(\mathbf{e}_h,\boldsymbol{s}_h,\mathbf{e}_h)$$

$$= ((\mathbf{e}_h\cdot\nabla)\boldsymbol{s}_h,\mathbf{e}_h) + \frac{1}{2}(\nabla\cdot\mathbf{e}_h,\boldsymbol{s}_h\cdot\mathbf{e}_h)$$

$$\le \left(\|\nabla\boldsymbol{s}_h\|_\infty + \frac{\|\boldsymbol{s}_h\|_\infty^2}{4\mu}\right)\|\mathbf{e}_h\|^2 + \frac{\mu}{4}\|\nabla\cdot\mathbf{e}_h\|^2.$$

$$\|\mathbf{e}_h(t)\|^2 \le e^{Lt}\Bigg(\|\mathbf{e}_h(0)\|^2 + \underbrace{\frac{2}{L}\int_0^t \|\partial_t\boldsymbol{\varepsilon}_h\|^2\,dt}_{O(h^{2(k+1)})} + \underbrace{\frac{2}{\mu}\int_0^t \|p-\pi_h p\|^2\,dt}_{O(h^{2k})}.$$

$$+ \underbrace{\frac{C^2}{L}\int_0^t \|\nabla\boldsymbol{\varepsilon}_h\|^2\,dt}_{O(h^{2k})}\Bigg).$$



Galerkin method with grad-div P_2/P_1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega))}$
h^2
h^3
$\nu = 10^{-4}$
$\nu = 10^{-6}$
$\nu = 10^{-8}$
3
4
5
6
7
level

The Navier-Stokes equations: Error bounds of Order k

Inf-sup stable elements.

div-free: Schroeder & Lube (2017)

grad-div: de Frutos, G-A, John & Novo (2018)

LPS: Ahmed & Matthies (2021) (to appear).

Non Inf-sup stable elements.

LPS: de Frutos, G-A, John & Novo (2019)

SPS: G-A, John & Novo (2021).

The Navier-Stokes equations: Methos of order $k + 1/2$

Non inf-sup stable elements of equal order. $O(h^{k+1/2})$ error.

Hansbo & Szepessy (1990). Space-time linear elements, only $O(h^{3/2})$.

Burman & Fernández (2007). Continuous Interior Penalty method.

Cheng, Feng & Zhou (2019). Two-level LPS

de Frutos, G-A, John & Novo (2019). LPS term by term stabilization.

H(div)-conforming methods (discontinuous elements). $O(h^{k+1/2})$ error.

Han & Hou (2021). Raviart-Thomas, DBM, upwind stab. term