Robust discretizations and solvers for poroelastic problems

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Outline

- Introduction to Poroelasticity problem
- Solvers for Poroelastic problems. Robust Preconditioners
- Parameter-robust preconditioners for the full stabilized P1-RT0-P0 system
- Cheaper approach: special static condensation and parameter-robust preconditioners
- Conclusions

Poroelasticity problem. Introduction

- The theory of poro-elasticity addresses the time dependent coupling between the deformation of a porous material and the fluid flow inside it.

Poroelasticity problem. Subsidence

SUBSIDENCE from groundwater pumping in San Joaquin Valley (California)

Courtesy of California Department of Water Resources

Poroelasticity problem. Subsidence

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Poroelasticity problem

QUASI-STATIC BIOT'S MODEL:

Equilibrium equation: $\operatorname{div} \boldsymbol{\sigma}' - \alpha \nabla p = \rho \mathbf{g}, \quad \text{in}\, \Omega,$

(or equivalently $\operatorname{div} \boldsymbol{\sigma} = \rho \mathbf{g},\ \boldsymbol{\sigma} = \boldsymbol{\sigma}' - \alpha \mathbf{I} p$)

Generalized form of Hooke's law: $\boldsymbol{\sigma}' = \lambda \operatorname{tr}(\boldsymbol{\epsilon})\mathbf{I} + 2\mu\boldsymbol{\epsilon}, \quad \text{in}\, \Omega,$

Compatibility equation: $\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^t), \quad \text{in}\, \Omega.$

Darcy's law: $\mathbf{w} = -\dfrac{1}{\mu_f} K(\nabla p - \rho_f \mathbf{g}), \quad \text{in}\, \Omega,$

Continuity equation: $\dfrac{\partial}{\partial t}\left(\dfrac{1}{M} p + \alpha \nabla \cdot \mathbf{u}\right) + \nabla \cdot \mathbf{w} = f, \quad \text{in}\, \Omega.$

- λ and μ: Lamé coefficients
- α: Biot-Willis constant and M: Biot's modulus
- K: Permeability of the porous medium and ρ: density of the solid
- μ_f: viscosity of the fluid and ρ_f: density of the fluid
- $\mathbf{u}$: displacement vector and p: pore pressure
- $\boldsymbol{\sigma}'$ and $\boldsymbol{\epsilon}$: effective stress and strain tensors
- $\mathbf{w}$: velocity of the fluid relative to the soil
- f: a forced fluid extraction or injection process and $\mathbf{g}$: gravity vector

Poroelasticity problem

Two-field (displacement-pressure) formulation

$$
\begin{aligned}
&-\nabla(\lambda+\mu)\nabla\cdot\mathbf{u}-\nabla\cdot\mu\nabla\mathbf{u}+\alpha\,\nabla\,p=\rho\mathbf{g},\\
&\frac{1}{M}\frac{\partial p}{\partial t}+\alpha\frac{\partial}{\partial t}(\nabla\cdot\mathbf{u})-\nabla\cdot\left(\frac{1}{\mu_f}K(\nabla p-\rho_f\mathbf{g})\right)=f.
\end{aligned}
$$

Three-field (fluid velocity) formulation

$$
\begin{aligned}
&-\nabla(\lambda+\mu)\nabla\cdot\mathbf{u}-\nabla\cdot\mu\nabla\mathbf{u}+\alpha\,\nabla\,p=\rho\mathbf{g},\\
&K^{-1}\mu_f\mathbf{w}+\nabla p=\rho_f\mathbf{g},\\
&\frac{1}{M}\frac{\partial p}{\partial t}+\alpha\frac{\partial}{\partial t}(\nabla\cdot\mathbf{u})+\nabla\cdot\mathbf{w}=f.
\end{aligned}
$$

Poroelasticity problem

Three-field (solid pressure) formulation

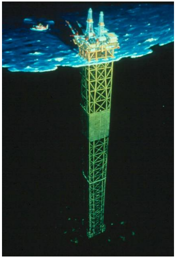
$$
\begin{aligned}
&-\nabla \mu \nabla \cdot \mathbf{u} - \nabla \cdot \mu \nabla \mathbf{u} + \nabla p_s + \alpha \nabla p = \rho \mathbf{g}, \\
&\lambda^{-1} p_s + \nabla \cdot \mathbf{u} = 0, \\
&\frac{1}{M}\frac{\partial p}{\partial t} + \alpha \frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) - \nabla \cdot \left(\frac{1}{\mu_f} K (\nabla p - \rho_f \mathbf{g}) \right) = f.
\end{aligned}
$$

Three-field (total pressure) formulation (Lee, Mardal, Winther, 2017)

$$
\begin{aligned}
&-\nabla \mu \nabla \cdot \mathbf{u} - \nabla \cdot \mu \nabla \mathbf{u} + \nabla p_T = \rho \mathbf{g}, \\
&-\nabla \cdot \mathbf{u} - \lambda^{-1}(p_T - \alpha p) = 0, \\
&\left(\frac{1}{M} + \frac{\alpha^2}{\lambda} \right) \frac{\partial p}{\partial t} - \frac{\alpha}{\lambda}\frac{\partial p_T}{\partial t} - \nabla \cdot \left(\frac{1}{\mu_f} K (\nabla p - \rho_f \mathbf{g}) \right) = f.
\end{aligned}
$$

Poroelasticity problem - Many Applications!

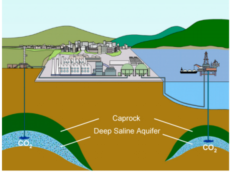
Reservoir Engineering



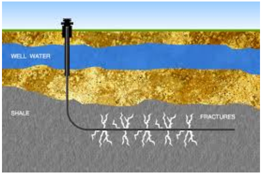
Bioengineering

Earthquake Engineering

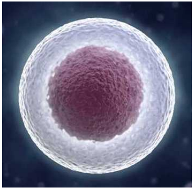
Carbon Dioxide Storage



Hydraulic Fracturing

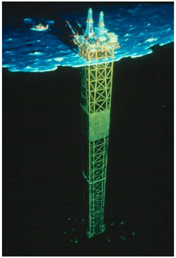


Animal Cells



Poroelasticity problem - Many Applications!

Reservoir Engineering

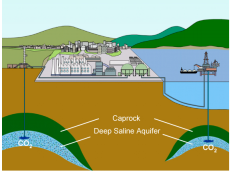


Bioengineering

Earthquake Engineering

Large variation of model parameters in many practical problems !!

Carbon Dioxide Storage



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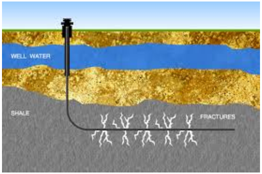

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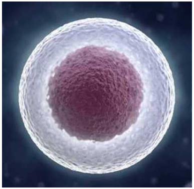

Poroelasticity problem. Discretization schemes

- Finite Difference schemes

  - F.J. Gaspar, F.J. Lisbona, P.N. Vabishchevich, A Finite Difference Analysis of Biot's Consolidation Model. Applied Numerical Mathematics, 44 (2003) 487-506.

  - F.J. Gaspar, F.J. Lisbona, P.N. Vabischevich, Staggered grid discretizations for the quasi-static Biot's consolidation problem, Applied Numerical Mathematics 56 (2006) pp. 888-898.

- Finite Volume methods

  - R.E. Ewing, O.P. Iliev, R.D. Lazarov, and A. Naumovich, On convergence of certain finite volume difference discretizations for 1-D poroelasticity interface problems, Numerical Methods for Partial Differential Equations 23 (3) (2007), 652-671.

  - J. M. Nordbotten, Stable cell-centered finite volume discretization for Biot equations, SIAM Journal on Numerical Analysis 54 (2) (2016) 942-968.

- Finite Element discretizations

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- Finite Element discretizations

DESIRABLE PROPERTIES

- Free of non-physical oscillations (MONOTONICITY?)
- Uniform stability with respect to the discretization and physical parameters

Poroelasticity problem. Robust discretization schemes

Search for parameter-robust stable discretizations

- J.J. Lee, Robust error analysis of coupled mixed methods for Biot's consolidation model, Journal of Scientific Computing 69 (2016) 610-632.
- J.J. Lee, K.-A. Mardal, and R. Winther. Parameter-robust discretization and preconditioning of Biot's consolidation model. SIAM Journal on Scientific Computing, 39 (2017) A1-A24.
- J. Adler, F.j. Gaspar, X. Hu, C. Rodrigo, L.T. Zikatanov, Robust Block Preconditioners for Biot's Model, Domain Decomposition Methods in Science and Engineering XXIV in Lecture Notes in Computational Science and Engineering, Vol. 125, Bjostad, P.E., Brenner, S.C., Halpern, L., Kim, H.H., Kornhuber, R., Rahman, T., Widlund, O.B. (Eds.), 2018.
- Q. Hong, J. Kraus, Parameter-robust stability of classical three-field formulation of Biot's consolidation model, ETNA, 48 (2018) 202-226.

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After discretization... we need to solve $\mathcal{A}\mathbf{x} = \mathbf{f}$

- Strongly coupled, ill-conditioned large systems of equations.
- Many physical parameters $(\lambda, \mu, M, K, \alpha)$ and discretization parameters (h, τ) are involved.

Solution of large-sparse systems

DESIRABLE PROPERTIES

- Robust convergence with respect to the discretization and physical parameters.
- Computationally efficient.

Solution of large-sparse systems

DESIRABLE PROPERTIES

- Robust convergence with respect to the discretization and physical parameters.
- Computationally efficient.

Mainly two approaches:

- Iterative coupling methods: solve sequentially the equations for fluid flow and geomechanics until a converged solution is achieved.
  - Flexibility: two different codes for fluid flow and geomechanics can be linked for solving the poroelastic problems.
  - Most frequently used: fixed-stress split method.

    J. Kim, H.A. Tchelepi, R. Juanes, Stability, Accuracy, and Efficiency of Sequential Methods for Coupled Flow and Geomechanics. Society of Petroleum Engineers (2011)

    A. Mikelic, M.F. Wheeler, Convergence of iterative coupling for coupled flow and geomechanics, Comput. Geosci. (2013)

    J. Both, M. Borregales, J.M. Nordbotten, K. Kumar, F. Radu, Robust fixed stress splitting for Biot's equations in heterogeneous media, Applied Mathematics Letters. (2017)
- Monolithic or fully coupled methods: the linear system is solved simultaneously for all the unknowns.

MONOLITHIC APPROACHES

- Monolithic multigrid methods (design of the smoother)
  - F.J. Gaspar, F.J. Lisbona, C. Oosterlee, R. Wienands, A systematic comparison of coupled and distributive smoothing in multigrid for the poroelasticity system, Numer. Linear Algebra Appl. 11 (2004) 93-113.
  - P. Luo, C. Rodrigo, F.J. Gaspar, C.W. Oosterlee, On an Uzawa smoother in multigrid for poroelasticity equations, Numer. Linear Algebra Appl. 24 (1) (2017). http://dx.doi.org/10.1002/nla.2074. e2074 nla.2074.
  - F.J. Gaspar, C.Rodrigo, On the fixed-stress split scheme as smoother in multigrid methods for coupling flow and geomechanics, Comput. Methods Appl. Mech. Engrg. 326 (2017) 526–540
- Preconditioners for Krylov subspace methods
  - L. Bergamaschi, M. Ferronato, G. Gambolati, Novel preconditioners for the iterative solution to FE-discretized coupled consolidation equations, Comput. Methods Appl. Mech. Engrg. 196 (25) (2007) 2647-2656.
  - M. Ferronato, L. Bergamaschi, G. Gambolati, Performance and robustness of block constraint preconditioners in finite element coupled consolidation problems, Internat. J. Numer. Methods Engrg. 81 (2010) 381-402.
  - N. Castelleto, J.A. White, H.A. Tchelepi, Accuracy and convergence properties of the fixed-stress iterative solution of two-way coupled poromechanics, Int. J. Numer. Anal. Methods Geomech. 39 (2015) 1593-1618.
  - J.J. Lee, K.-A. Mardal, and R. Winther, Parameter-robust discretization and preconditioning of Biot's consolidation model. SIAM Journal on Scientific Computing, 39 (2017) A1-A24.

Preconditioner Framework

Setup:

- Hilbert space $\mathcal{H}$ equipped with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$
- Operator $\mathcal{A} : \mathcal{H} \mapsto \mathcal{H}'$

Linear system: given $\mathbf{f} \in \mathcal{H}'$, find $\mathbf{x} \in \mathcal{H}$ such that $\mathcal{A}\mathbf{x} = \mathbf{f}$

Well-posedness:

Continuity: $$\sup_{\mathbf{0}\neq\mathbf{x}\in\mathcal{H}} \sup_{\mathbf{0}\neq\mathbf{y}\in\mathcal{H}} \frac{\langle \mathcal{A}\mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}}} \leq \beta$$

Inf-sup condition: $$\inf_{\mathbf{0}\neq\mathbf{x}\in\mathcal{H}} \sup_{\mathbf{0}\neq\mathbf{y}\in\mathcal{H}} \frac{\langle \mathcal{A}\mathbf{x}, \mathbf{y}\rangle}{\|\mathbf{x}\|_{\mathcal{H}}\|\mathbf{y}\|_{\mathcal{H}}} \geq \gamma > 0$$

The well-posedness of the discretized system provides a convenient framework with which to construct block preconditioners

D. Loghin and A. Wathen, Analysis of preconditioners for saddle-point problems, SIAM Journal on Scientific Computing, 25 (2004), 2029-2049.

K.-A. Mardal and R. Winther, Preconditioning discretizations of systems of partial differential equations, Numerical Linear Algebra with Applications, 18 (2011), 1-40.

Norm-equivalent Preconditioner

Preconditioner $\mathcal{M} : \mathcal{H}' \mapsto \mathcal{H}$ is symmetric positive definite $(\mathcal{M}\mathcal{A} : \mathcal{H} \mapsto \mathcal{H})$

Norm-equivalence

If $\mathcal{A}$ is well-posed w.r.t the norm $\|\cdot\|_{\mathcal{H}}$. Choose $\mathcal{M}$ such that

$$c_1\|\mathbf{x}\|_{\mathcal{H}}^2 \leq \|\mathbf{x}\|_{\mathcal{M}^{-1}}^2 \leq c_2\|\mathbf{x}\|_{\mathcal{H}}^2$$

then $\mathcal{M}$ and $\mathcal{A}$ are norm-equivalent and $\kappa(\mathcal{M}\mathcal{A}) \leq \frac{c_2\beta}{c_1\gamma}$

Theorem (Convergence of preconditioned MINRES)

There exists a constant $\delta \in (0,1)$ such that

$$\langle \mathcal{M}\mathcal{A}(\mathbf{x}-\mathbf{x}^m), \mathcal{A}(\mathbf{x}-\mathbf{x}^m)\rangle^{1/2} \leq 2\delta^m \langle \mathcal{M}\mathcal{A}(\mathbf{x}-\mathbf{x}^0), \mathcal{A}(\mathbf{x}-\mathbf{x}^0)\rangle^{1/2}$$

where

$$\delta = \frac{\kappa(\mathcal{M}\mathcal{A})-1}{\kappa(\mathcal{M}\mathcal{A})+1}$$

Field-of-Values Equivalent Preconditioner

Preconditioner $\mathcal{M}_L : \mathcal{H}' \mapsto \mathcal{H}$ is a general operator ($\mathcal{M}_L\mathcal{A} : \mathcal{H} \mapsto \mathcal{H}$)

Field-of-Value equivalence

The operators $\mathcal{M}_L$ and $\mathcal{A}$ are FoV-equivalent, if for any $\mathbf{x} \in \mathcal{H}$,

$$\sigma \leq \frac{(\mathcal{M}_L\mathcal{A}\mathbf{x}, \mathbf{x})_{\mathcal{M}^{-1}}}{(\mathbf{x}, \mathbf{x})_{\mathcal{M}^{-1}}}, \quad \frac{\|\mathcal{M}_L\mathcal{A}\mathbf{x}\|_{\mathcal{M}^{-1}}}{\|\mathbf{x}\|_{\mathcal{M}^{-1}}} \leq \Upsilon.$$

Theorem (Convergence of preconditioned GMRES)

If $\mathbf{x}^m$ is the m-th iteration of GMRES method and $\mathbf{x}$ is the exact solution,

$$\frac{\|\mathcal{M}_L\mathcal{A}(\mathbf{x} - \mathbf{x}^m)\|_{\mathcal{M}^{-1}}}{\|\mathcal{M}_L\mathcal{A}(\mathbf{x} - \mathbf{x}^0)\|_{\mathcal{M}^{-1}}} \leq \left(1 - \frac{\sigma^2}{\Upsilon^2}\right)^{m/2}$$

Two-field formulation

We consider the discretization of poroelasticity problem given by operators of the form $\mathcal{A}_C = \begin{pmatrix} A & B' \\ B & -C \end{pmatrix}$, where C is bounded, selfadjoint and positive definite.

$\mathcal{A}_C$ is an isomorphism $\Leftrightarrow$ For any $q \in \mathcal{Q}_h^{k'}$, $\sup\limits_{v \in \mathcal{U}_h^k} \dfrac{\langle B\, v, q \rangle}{\| \, v \, \|_A} \geq \gamma_B \, \| \, q \, \| - \| \, q \, \|_C$

If inf-sup condition for B is satisfied with $C = 0$,
then it is also satisfied with $C > 0$

Stable finite element pair for Stokes is also stable for poroelasticity

C. Rodrigo, F.J. Gaspar, X. Hu, L.T. Zikatanov, Stability and monotonicity for some discretizations of the Biot's consolidation model, Computer methods in applied mechanics and engineering, 2016

- P1-P1 + Stabilization
- MINI element + Stabilization

Robust preconditioners for the two-field formulation in:

J. Adler, F.j. Gaspar, X. Hu, C. Rodrigo, L.T. Zikatanov, Robust Block Preconditioners for Biot's Model, Domain Decomposition Methods in Science and Engineering XXIV in Lecture Notes in Computational Science and Engineering, Vol. 125, Bjostad, P.E., Brenner, S.C., Halpern, L., Kim, H.H., Kornhuber, R., Rahman, T., Widlund, O.B. (Eds.), 2018.

Three-field formulation

$$\boxed{\begin{aligned} &-\operatorname{div}\boldsymbol{\sigma}' + \alpha\nabla p = \rho\boldsymbol{g}, \qquad \text{where} \quad \boldsymbol{\sigma}' = 2\mu\varepsilon(\boldsymbol{u}) + \lambda\operatorname{div}(\boldsymbol{u})\boldsymbol{I}, \\ &\mathbf{K}^{-1}\mu_f\boldsymbol{w} + \nabla p = \rho_f\boldsymbol{g}, \\ &\frac{\partial}{\partial t}\left(\frac{1}{M}p + \alpha\operatorname{div}\boldsymbol{u}\right) + \operatorname{div}\boldsymbol{w} = f. \end{aligned}}$$

Boundary conditions:

$$p = 0, \quad \text{for} \quad x \in \Gamma_t, \quad \boldsymbol{\sigma}'\boldsymbol{n} = \mathbf{0}, \quad \text{for} \quad x \in \Gamma_t,$$
$$\boldsymbol{u} = \mathbf{0}, \quad \text{for} \quad x \in \Gamma_c, \quad \boldsymbol{w}\cdot\boldsymbol{n} = 0, \quad \text{for} \quad x \in \Gamma_c,$$

Initial condition:

$$\left(\frac{1}{M}p + \alpha\operatorname{div}\mathbf{u}\right)(\mathbf{x}, 0) = 0$$

- $\Gamma_t \cup \Gamma_c = \Gamma$.
- $\mathbf{n}$ is the unit outward normal to the boundary.

Variational formulation

Introducing the spaces:

$$\begin{aligned}
\mathbf{V} &= \left\{\mathbf{u} \in \boldsymbol{H}^1(\Omega), \mathbf{u} = \mathbf{0} \text{ en } \Gamma_c\right\}, \\
\boldsymbol{W} &= \{\boldsymbol{w} \in \boldsymbol{H}(\mathrm{div}, \Omega), (\boldsymbol{w} \cdot \boldsymbol{n})|_{\Gamma_c} = 0\}, \\
Q &= L^2(\Omega),
\end{aligned}$$

and the bilinear form:

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \int_\Omega \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v})\, \mathrm{d}\Omega + \lambda \int_\Omega \mathrm{div}\, \boldsymbol{u}\, \mathrm{div}\, \boldsymbol{v}\, \mathrm{d}\Omega.$$

For each $t \in (0, T]$, find $(\boldsymbol{u}(t), \boldsymbol{w}(t), p(t)) \in \boldsymbol{V} \times \boldsymbol{W} \times Q$ such that

$$\begin{aligned}
& a(\boldsymbol{u}, \boldsymbol{v}) - (\alpha p, \mathrm{div}\, \boldsymbol{v}) = (\rho \boldsymbol{g}, \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\
& (\boldsymbol{K}^{-1} \mu_f \boldsymbol{w}, \boldsymbol{r}) - (p, \mathrm{div}\, \boldsymbol{r}) = (\rho_f \boldsymbol{g}, \boldsymbol{r}), \quad \forall\, \boldsymbol{r} \in \boldsymbol{W}, \\
& \left(\frac{1}{M}\frac{\partial p}{\partial t}, q\right) + \left(\alpha\, \mathrm{div}\, \frac{\partial \boldsymbol{u}}{\partial t}, q\right) + (\mathrm{div}\, \boldsymbol{w}, q) = (f, q), \quad \forall\, q \in Q.
\end{aligned}$$

Finite element discretization

- Implicit Euler scheme + triple of finite element spaces $(\boldsymbol{V}_h, \boldsymbol{W}_h, Q_h)$

For $m \geq 1$, find $(\boldsymbol{u}_h^m, \boldsymbol{w}_h^m, p_h^m) \in \boldsymbol{V}_h \times \boldsymbol{W}_h \times Q_h$ such that

$$a(\boldsymbol{u}_h^m, \boldsymbol{v}_h) - (\alpha p_h^m, \operatorname{div} \boldsymbol{v}_h) = (\rho \boldsymbol{g}, \boldsymbol{v}_h), \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\tau(\boldsymbol{K}^{-1}\mu_f \boldsymbol{w}_h^m, \boldsymbol{r}_h) - \tau(p_h^m, \operatorname{div} \boldsymbol{r}_h) = \tau(\rho_f \boldsymbol{g}, \boldsymbol{r}_h), \quad \forall\ \boldsymbol{r}_h \in \boldsymbol{W}_h,$$

$$\left(\frac{1}{M} p_h^m, q_h\right) + (\alpha \operatorname{div} \boldsymbol{u}_h^m, q_h) + \tau(\operatorname{div} \boldsymbol{w}_h^m, q_h) = (\widetilde{f}, q_h), \quad \forall\ q_h \in Q_h.$$

Stability

Definition (Adler, Gaspar, Hu, Ohm, Rodrigo, & Zikatanov 2018)

The triple of spaces $(\boldsymbol{V}_h, \boldsymbol{W}_h, Q_h)$ is *Stokes-Biot stable* if and only if the following conditions are satisfied:

- $a(\boldsymbol{u}_h, \boldsymbol{v}_h) \leq C_{\boldsymbol{V}} \|\boldsymbol{u}_h\|_1 \|\boldsymbol{v}_h\|_1$, for all $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$;
- $a(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \alpha_{\boldsymbol{V}} \|\boldsymbol{u}_h\|_1^2$, for all $\boldsymbol{u}_h \in \boldsymbol{V}_h$;
- The pair of spaces $(\boldsymbol{W}_h, Q_h)$ is Poisson stable;
- The pair of spaces $(\boldsymbol{V}_h, Q_h)$ is Stokes stable.

Stability

We introduce the weighted norm:

$$|||(\boldsymbol{u}_h, \boldsymbol{w}_h, p_h)||| \quad = \quad \Bigg(\|\boldsymbol{u}_h\|_A^2 + \tau \|\boldsymbol{w}_h\|_{K^{-1}\mu_f}^2 + \tau^2 \left(\frac{\alpha^2}{\xi^2} + \frac{1}{M} \right)^{-1} \|\operatorname{div} \boldsymbol{w}_h\|^2 + \left(\frac{\alpha^2}{\xi^2} + \frac{1}{M} \right) \|p_h\|^2 \Bigg)^{1/2}, \ \xi = \sqrt{\lambda + \frac{2\mu}{d}}.$$

Theorem (Adler, Gaspar, Hu, Ohm, Rodrigo, & Zikatanov 2018)

If the triple $(\boldsymbol{V}_h, \boldsymbol{W}_h, Q_h)$ is Stokes-Biot stable, the block matrix form $\mathcal{A}$ is well-posed with respect to the weighted norm, i.e. the following continuity and inf-sup condition hold for $\mathbf{x}_h = (\mathbf{u}_h, \mathbf{w}_h, p_h) \in \mathbf{X}_h = \mathbf{V}_h \times \mathbf{W}_h \times Q_h$ and $\mathbf{y}_h \in \mathbf{X}_h$,

$$\sup_{0 \neq \boldsymbol{x}_h \in \boldsymbol{X}_h} \sup_{0 \neq \boldsymbol{y}_h \in \boldsymbol{X}_h} \frac{(\mathcal{A}\boldsymbol{x}_h, \boldsymbol{y}_h)}{|||\boldsymbol{x}_h|||\, |||\boldsymbol{y}_h|||} \leq \varsigma,$$

$$\inf_{0 \neq \boldsymbol{y}_h \in \boldsymbol{X}_h} \sup_{0 \neq \boldsymbol{x}_h \in \boldsymbol{X}_h} \frac{(\mathcal{A}\boldsymbol{x}_h, \boldsymbol{y}_h)}{|||\boldsymbol{x}_h|||\, |||\boldsymbol{y}_h|||} \geq \gamma,$$

with constants $\varsigma > 0$ and $\gamma > 0$ independent of mesh size h, time step size τ, and the physical parameters.

Finite element discretization: P1-RT0-P0

Implicit Euler scheme + P1-RT0-P0

$$\boldsymbol{V}_h = \left\{ \boldsymbol{v}_h \in \boldsymbol{V} \mid \boldsymbol{v}_h|_T \in [\mathbb{P}_1(T)]^d, \text{ for all } T \in \mathcal{T}_h \right\},$$

$$\boldsymbol{W}_h = \left\{ \boldsymbol{w}_h \in \boldsymbol{W} \mid \boldsymbol{w}_h|_T = \boldsymbol{a} + \eta \mathbf{x}, \ \boldsymbol{a} \in \mathbb{R}^d, \ \eta \in \mathbb{R}, \ \ \forall\, T \in \mathcal{T}_h \right\},$$

$$Q_h = \{ q_h \in Q \mid q_h|_T \in \mathbb{P}_0(T), \ \ \forall\, T \in \mathcal{T}_h \}.$$

Finite element discretization: P1-RT0-P0

Implicit Euler scheme + P1-RT0-P0

$$\boldsymbol{V}_h = \left\{ \boldsymbol{v}_h \in \boldsymbol{V} \mid \boldsymbol{v}_h|_T \in [\mathbb{P}_1(T)]^d, \text{ for all } T \in \mathcal{T}_h \right\},$$

$$\boldsymbol{W}_h = \left\{ \boldsymbol{w}_h \in \boldsymbol{W} \mid \boldsymbol{w}_h|_T = \boldsymbol{a} + \eta \mathbf{x}, \; \boldsymbol{a} \in \mathbb{R}^d, \; \eta \in \mathbb{R}, \; \forall\, T \in \mathcal{T}_h \right\},$$

$$Q_h = \{ q_h \in Q \mid q_h|_T \in \mathbb{P}_0(T), \; \forall\, T \in \mathcal{T}_h \}.$$

- RT0-P0 is Poisson stable
- P1-P0 is not Stokes stable

Numerical Experiment

- Domain: $\Omega = (0,1) \times (0,1)$.
- Dirichlet boundary conditions for displacements and pressure.
- We cover Ω with a uniform triangular grid by dividing an $(N \times N)$ uniform square mesh into right triangles.
- $\lambda = 2, \mu = 1$.
- $\mu_f = 1, \alpha = 1, M = 10^6$.
- Diagonal permeability tensor $\boldsymbol{K} = k\boldsymbol{I}$ with constant k.
- We set $\tau = 1$ and $t_{\max} = 1$, so that we only perform one time step.
- the exact solution is given by

$$
\begin{aligned}
\boldsymbol{u}(x,y,t) &= \operatorname{curl}\varphi = \begin{pmatrix} \partial_y \varphi \\ -\partial_x \varphi \end{pmatrix}, \quad \varphi(x,y) = [xy(1-x)(1-y)]^2, \\
p(x,y,t) &= 1.
\end{aligned}
$$

Energy norm and L^2-norm for displacement and pressure errors

		$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
$\kappa = 10^{-4}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_A$	0.0209	0.0089	0.0043	0.0022	0.0011
	$\|\|p - p_h\|\|_{L^2}$	0.0535	0.0088	0.0015	0.0003	7.38×10^{-5}
$\kappa = 10^{-6}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_A$	0.0477	0.0271	0.0060	0.0022	0.0011
	$\|\|p - p_h\|\|_{L^2}$	0.3277	0.3199	0.0763	0.0099	0.0012
$\kappa = 10^{-8}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_A$	0.0503	0.0497	0.0418	0.0147	0.0019
	$\|\|p - p_h\|\|_{L^2}$	0.3553	0.7157	1.1509	0.6537	0.1152
$\kappa = 10^{-10}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_A$	0.0503	0.0503	0.0501	0.0484	0.0330
	$\|\|p - p_h\|\|_{L^2}$	0.3550	0.7271	1.4576	2.7836	3.4508

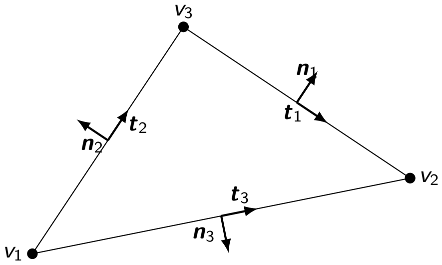
Results confirm poor approximation when k/h is small

Stabilization by Bubbles (Girault & Raviart 1986)

Enrich linear finite element by face bubbles: $\boldsymbol{V}_h = \boldsymbol{V}_{h,l} \oplus \boldsymbol{V}_b$

Example: bubbles in 2D

$$\Phi_1 = \boldsymbol{n}_1\lambda_2\lambda_3, \quad \Phi_2 = \boldsymbol{n}_2\lambda_1\lambda_3, \quad \Phi_3 = \boldsymbol{n}_3\lambda_1\lambda_2$$



P1+bubbles-RT0-P0 is Stokes-Biot stable

Stabilization by face bubbles.

- Implicit Euler scheme + $\boldsymbol{V}_h$-RT0-P0

$$\boldsymbol{V}_h = \boldsymbol{V}_{h,l} \oplus \boldsymbol{V}_b, \text{ where}$$

$$\boldsymbol{V}_{h,l} = \{\boldsymbol{v}_h \in \boldsymbol{V} \mid \boldsymbol{v}_h|_T \in [\mathbb{P}_1(T)]^d, \text{ for all } T \in \mathcal{T}_h\},$$

$$\boldsymbol{V}_b = \text{span}\{\boldsymbol{\Phi}_e\}_{e\in\mathcal{E}},$$

$$\boldsymbol{W}_h = \{\boldsymbol{w}_h \in \boldsymbol{W} \mid \boldsymbol{w}_h|_T = \boldsymbol{a} + \eta \mathbf{x}, \ \boldsymbol{a} \in \mathbb{R}^d, \ \eta \in \mathbb{R}, \ \ \forall\, T \in \mathcal{T}_h\},$$

$$Q_h = \{q_h \in Q \mid q_h|_T \in \mathbb{P}_0(T), \ \ \forall\, T \in \mathcal{T}_h\}.$$

For $m \geq 1$, find $(\boldsymbol{u}_h^m, \boldsymbol{w}_h^m, p_h^m) \in \boldsymbol{V}_h \times \boldsymbol{W}_h \times Q_h$ such that

$$a(\boldsymbol{u}_h^m, \boldsymbol{v}_h) - (\alpha p_h^m, \operatorname{div} \boldsymbol{v}_h) = (\rho \boldsymbol{g}, \boldsymbol{v}_h), \quad \forall\ \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

$$\tau(\boldsymbol{K}^{-1}\mu_f \boldsymbol{w}_h^m, \boldsymbol{r}_h) - \tau(p_h^m, \operatorname{div} \boldsymbol{r}_h) = \tau(\rho_f \boldsymbol{g}, \boldsymbol{r}_h), \quad \forall\ \boldsymbol{r}_h \in \boldsymbol{W}_h,$$

$$\left(\frac{1}{M} p_h^m, q_h\right) + (\alpha \operatorname{div} \boldsymbol{u}_h^m, q_h) + \tau(\operatorname{div} \boldsymbol{w}_h^m, q_h) = (\widetilde{f}, q_h), \quad \forall\ q_h \in Q_h.$$

Stabilization by face bubbles

We have the following block form of the discrete problem:

$$\boxed{\mathcal{A}\begin{pmatrix} \boldsymbol{U}_b \\ \boldsymbol{U}_l \\ \boldsymbol{P} \\ \boldsymbol{W} \end{pmatrix} = \boldsymbol{b}, \quad \text{with} \quad \mathcal{A} = \begin{pmatrix} A_{bb} & A_{bl} & \alpha B_b^T & 0 \\ A_{bl}^T & A_{ll} & \alpha B_l^T & 0 \\ -\alpha B_b & -\alpha B_l & \frac{1}{M} M_p & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} \end{pmatrix}}$$

$$a(\boldsymbol{u}_h^b, \boldsymbol{v}_h^b) \to A_{bb}, \quad a(\boldsymbol{u}_h^l, \boldsymbol{v}_h^b) \to A_{bl}, \quad a(\boldsymbol{u}_h^l, \boldsymbol{v}_h^l) \to A_{ll},$$
$$-(p_h, \operatorname{div} \boldsymbol{v}_h^b) \to B_b, \quad -(p_h, \operatorname{div} \boldsymbol{v}_h^l) \to B_l, \quad -(p_h, \operatorname{div} \boldsymbol{r}_h) \to B_{\boldsymbol{w}},$$
$$(K^{-1}\mu_f \boldsymbol{w}_h, \boldsymbol{r}_h) \to M_w, \quad (p_h, q_h) \to M_p,$$

We also denote:

$$a(\mathbf{u}_h, \mathbf{v}_h) \to A_{\boldsymbol{u}}, \quad -(\operatorname{div} \mathbf{u}_h, q_h) \to B_{\boldsymbol{u}},$$

such that $A_{\boldsymbol{u}} = \begin{pmatrix} A_{bb} & A_{bl} \\ A_{lb} & A_{ll} \end{pmatrix}$ and $B_{\boldsymbol{u}} = (B_b, B_l)$.

Numerical results. Poroelastic problem 2D

Energy norm and L^2-norm for displacement and pressure errors

		$N=8$	$N=16$	$N=32$	$N=64$	$N=128$
$k=10^{-4}$	$\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|_A$	0.0151	0.0072	0.0037	0.0019	0.0010
	$\|\|p-p_h\|\|_{L^2}$	0.0322	0.0168	0.0104	0.0052	0.0020
$k=10^{-6}$	$\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|_A$	0.0153	0.0073	0.0036	0.0018	0.0009
	$\|\|p-p_h\|\|_{L^2}$	0.0349	0.0161	0.0074	0.0032	0.0012
$k=10^{-8}$	$\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|_A$	0.0153	0.0073	0.0036	0.0018	0.0009
	$\|\|p-p_h\|\|_{L^2}$	0.0349	0.0162	0.0074	0.0035	0.0017
$k=10^{-10}$	$\|\|\boldsymbol{u}-\boldsymbol{u}_h\|\|_A$	0.0153	0.0073	0.0036	0.0018	0.0009
	$\|\|p-p_h\|\|_{L^2}$	0.0349	0.0162	0.0075	0.0035	0.0017

The errors are appropriately reduced independently of the physical parameters.

Stabilized P1-RT0-P0

We have the following block form of the discrete problem:

$$\mathcal{A}\begin{pmatrix} \boldsymbol{U}_b \\ \boldsymbol{U}_l \\ \boldsymbol{P} \\ \boldsymbol{W} \end{pmatrix} = \boldsymbol{b}, \quad \text{with} \quad \mathcal{A} = \begin{pmatrix} A_{bb} & A_{bl} & \alpha B_b^T & 0 \\ A_{bl}^T & A_{ll} & \alpha B_l^T & 0 \\ -\alpha B_b & -\alpha B_l & \frac{1}{M} M_p & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} \end{pmatrix}$$

Since the triple $(\boldsymbol{V}_h, \boldsymbol{W}_h, Q_h)$ is Stokes-Biot stable, the block matrix form $\mathcal{A}$ is well-posed with respect to the weighted norm:

$$|||(\boldsymbol{u}_h, \boldsymbol{w}_h, p_h)||| = \left(\|\boldsymbol{u}_h\|_A^2 + \tau \|\boldsymbol{w}_h\|_{K^{-1}\mu_f}^2 + \tau^2 \left(\frac{\alpha^2}{\xi^2} + \frac{1}{M} \right)^{-1} \|\operatorname{div} \boldsymbol{w}_h\|^2 + \left(\frac{\alpha^2}{\xi^2} + \frac{1}{M} \right) \|p_h\|^2 \right)^{1/2}, \ \xi = \sqrt{\lambda + \frac{2\mu}{d}}.$$

independently of discretization and physical parameters.

Norm-equivalent Preconditioner for the full system

Block diagonal preconditioners:

$$\mathcal{B}_D = \begin{pmatrix} A_{\boldsymbol{u}} & 0 & 0 \\ 0 & \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right) M_p & 0 \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

where $A_{\boldsymbol{w}} = B_{\boldsymbol{w}}^T M_p^{-1} B_{\boldsymbol{w}}$.

Norm-equivalent Preconditioner for the full system

Block diagonal preconditioners:

$$\mathcal{B}_D = \begin{pmatrix} A_{\boldsymbol{u}} & 0 & 0 \\ 0 & \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right) M_p & 0 \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

where $A_{\boldsymbol{w}} = B_{\boldsymbol{w}}^T M_p^{-1} B_{\boldsymbol{w}}$.

Inexact block diagonal preconditioners:

$$\widehat{\mathcal{B}_D} = \begin{pmatrix} Q_{\boldsymbol{u}} & 0 & 0 \\ 0 & Q_p & 0 \\ 0 & 0 & Q_{\boldsymbol{w}} \end{pmatrix}$$

where $Q_{\boldsymbol{u}}$: MG, Q_p: Jacobi, and $Q_{\boldsymbol{w}}$: Hiptmair-Xu/MG

Norm-equivalent Preconditioner for the full system

Block diagonal preconditioners:

$$\mathcal{B}_D = \begin{pmatrix} A_{\boldsymbol{u}} & 0 & 0 \\ 0 & \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right) M_p & 0 \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

where $A_{\boldsymbol{w}} = B_{\boldsymbol{w}}^T M_p^{-1} B_{\boldsymbol{w}}$.

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where $Q_{\boldsymbol{u}}$: MG, Q_p: Jacobi, and $Q_{\boldsymbol{w}}$: Hiptmair-Xu/MG

Theorem (Adler, Gaspar, Hu, Ohm, Rodrigo, & Zikatanov 2018)

If the linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$, then we have

$$\kappa(\mathcal{B}_D\mathcal{A}) \leq C_1 \text{ and } \kappa(\widehat{\mathcal{B}_D}\mathcal{A}) \leq C_2$$

where C_1 and C_2 are constants independent of the physical parameters (λ, μ, M, K, α) and the discretization parameters (h, τ).

FoV-equivalent Preconditioners for the full system

Block lower triangular preconditioner:

$$\mathcal{B}_L = \begin{pmatrix} A_{\boldsymbol{u}} & 0 & 0 \\ -\alpha B_{\boldsymbol{u}} & \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right) M_p & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

FoV-equivalent Preconditioners for the full system

Block lower triangular preconditioner:

$$\mathcal{B}_L = \begin{pmatrix} A_{\boldsymbol{u}} & 0 & 0 \\ -\alpha B_{\boldsymbol{u}} & \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right) M_p & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

Inexact block lower triangular preconditioner:

$$\widehat{\mathcal{B}_L} = \begin{pmatrix} Q_{\boldsymbol{u}}^{-1} & 0 & 0 \\ -\alpha B_{\boldsymbol{u}} & Q_p^{-1} & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & Q_{\boldsymbol{w}}^{-1} \end{pmatrix}^{-1}$$

Theorem (Adler, Gaspar, Hu, Ohm, Rodrigo, & Zikatanov 2018)

If the linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$, then we have

- $\mathcal{B}_L$ *and* $\mathcal{A}$ *are FoV-equivalent*
- $\widehat{\mathcal{B}_L}$ *and* $\mathcal{A}$ *are FoV-equivalent if* $\|I_{\boldsymbol{u}} - Q_{\boldsymbol{u}} A_{\boldsymbol{u}}\|_{A_{\boldsymbol{u}}} \leq \delta < 1$
- *FoV-equivalent constants are independent of the physical parameters* $(\lambda, \mu, M, K, \alpha)$ *and the discretization parameters* (h, τ)
- *Preconditioned GMRES converges uniformly*

FoV-equivalent Preconditioners for the full system

Block upper triangular preconditioner:

$$\mathcal{B}_U = \begin{pmatrix} A_{\boldsymbol{u}} & \alpha B_{\boldsymbol{u}}^T & 0 \\ 0 & \left(\frac{\alpha^2}{\lambda + \frac{2\mu}{d}} + \frac{1}{M}\right) M_p & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda + \frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

FoV-equivalent Preconditioners for the full system

Block upper triangular preconditioner:

$$\mathcal{B}_U = \begin{pmatrix} A_{\boldsymbol{u}} & \alpha B_{\boldsymbol{u}}^T & 0 \\ 0 & \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right) M_p & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \left(\frac{\alpha^2}{\lambda+\frac{2\mu}{d}} + \frac{1}{M}\right)^{-1} \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

Inexact block upper triangular preconditioner:

$$\widehat{\mathcal{B}_U} = \begin{pmatrix} Q_{\boldsymbol{u}}^{-1} & \alpha B_{\boldsymbol{u}}^T & 0 \\ 0 & Q_p^{-1} & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & Q_{\boldsymbol{w}}^{-1} \end{pmatrix}^{-1}$$

Theorem (Adler, Gaspar, Hu, Ohm, Rodrigo, & Zikatanov 2018)

If the linear system is well-posed w.r.t $\|\cdot\|_{\mathcal{H}}$, then we have

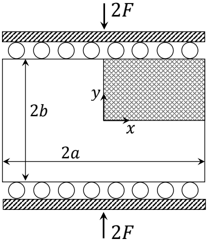
- *$\mathcal{B}_U$ and $\mathcal{A}$ are FoV-equivalent*
- *$\widehat{\mathcal{B}_U}$ and $\mathcal{A}$ are FoV-equivalent if $\|I_{\boldsymbol{u}} - Q_{\boldsymbol{u}} A_{\boldsymbol{u}}\|_{A_{\boldsymbol{u}}} \leq \delta < 1$*
- *FoV-equivalent constants are independent of the physical parameters $(\lambda, \mu, M, K, \alpha)$ and the discretization parameters (h, τ)*
- *Preconditioned GMRES converges uniformly*

Numerical experiments

- Linear system solved using preconditioned FGMRES to a relative residual tolerance of 10^{-8}
- $Q_{\boldsymbol{u}}$ is solved using AMG preconditioned GMRES to a relative residual tolerance of 10^{-3}
- $Q_{\boldsymbol{w}}$ is solved using HX preconditioned GMRES to a relative residual tolerance of 10^{-3}
- Q_p is calculated directly since the pressure block, $\mathcal{M}_p$, is diagonal
- Solved using HAZMATH: A Simple Finite Element, Graph, and Solver Library (www.hazmath.net)

Numerical experiments: Mandel problem

2D physical and computational domains for Mandel's problem



- Benchmark problem: the analytical solution for the pore pressure can be found in Abousleiman et al. 1996
- Computational domain: $\Omega = (0,1)^2$.
- Uniform triangular grid on Ω, obtained by dividing a $N \times N$ uniform square grid into right triangles.
- Material properties: $\mu_f = 1, \ \alpha = 1, \ M = 10^6$
- Lamé coefficients computed as $\lambda = \dfrac{E\nu}{(1-2\nu)(1+\nu)}$ and $\mu = \dfrac{E}{1+2\nu}$, where E is the Young modulus and ν is the Poisson ratio.

Iteration counts for the block preconditioners
with varying discretization parameters.

τ \ h	$\mathcal{B}_D$ $\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\mathcal{B}_L$ $\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\mathcal{B}_U$ $\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
0.1	28	35	37	38	37	15	17	17	17	16	15	17	17	17	17
0.01	21	22	28	33	35	10	12	15	16	16	9	12	14	16	16
0.001	19	19	19	22	27	8	8	9	12	14	7	7	8	11	13
0.0001	16	17	17	17	17	7	7	7	7	7	7	6	6	6	8

τ \ h	$\widehat{\mathcal{B}_D}$ $\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\widehat{\mathcal{B}_L}$ $\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\widehat{\mathcal{B}_U}$ $\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
0.1	28	35	38	38	37	15	17	17	17	16	15	17	18	17	17
0.01	21	22	28	33	35	10	12	15	16	16	9	12	14	16	16
0.001	19	19	19	22	27	8	8	10	12	14	7	7	9	11	14
0.0001	17	17	17	17	17	7	7	7	8	9	7	6	6	7	9

Iteration counts for the block preconditioners
with varying physical parameters K and ν

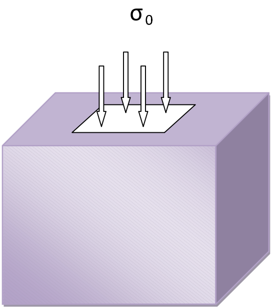
	$\nu = 0.0$ and varying K					
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$\mathcal{B}_D$	23	27	38	35	17	10
$\mathcal{B}_L$	7	9	15	16	9	5
$\mathcal{B}_U$	13	15	17	16	8	3
$\widehat{\mathcal{B}_D}$	35	29	38	35	17	10
$\widehat{\mathcal{B}_L}$	14	15	16	16	9	6
$\widehat{\mathcal{B}_U}$	27	19	17	16	9	2

	$K = 10^{-6}$ and varying ν					
	0.1	0.2	0.4	0.45	0.49	0.499
$\mathcal{B}_D$	45	50	39	37	32	21
$\mathcal{B}_L$	16	18	11	9	7	6
$\mathcal{B}_U$	20	22	15	13	11	12
$\widehat{\mathcal{B}_D}$	45	50	39	37	32	25
$\widehat{\mathcal{B}_L}$	17	19	12	10	11	9
$\widehat{\mathcal{B}_U}$	21	23	19	20	24	16

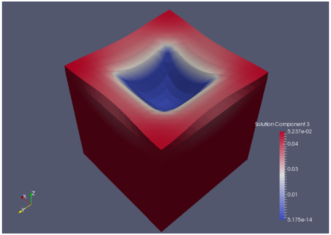
Numerical experiments: 3D footing problem

Three-dimensional footing problem

- Block of porous soil: $\Omega = (0,1)^3$
- Uniform load of intensity $\sigma_0 = 3 \times 10^4$ per unit area



Computational domain



Example solution

Numerical experiments: 3D footing problem

Iteration counts for the block preconditioners
with varying discretization parameters

$\tau \backslash h$	$\mathcal{B}_D$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\mathcal{B}_L$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\mathcal{B}_U$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
0.1	60	65	65	$*$	34	36	36	$*$	32	34	34	$*$
0.01	47	57	68	$*$	30	34	37	$*$	26	31	35	$*$
0.001	40	42	49	$*$	26	28	32	$*$	20	23	28	$*$
0.0001	40	42	42	$*$	24	35	36	$*$	20	20	21	$*$

$\tau \backslash h$	$\widehat{\mathcal{B}_D}$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\widehat{\mathcal{B}_L}$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\widehat{\mathcal{B}_U}$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
0.1	60	65	66	64	34	36	36	36	32	34	34	34
0.01	47	58	68	71	30	34	37	39	26	31	35	37
0.001	42	42	51	63	26	28	32	36	20	24	28	33
0.0001	40	42	42	45	24	25	27	29	21	22	23	25

($*$ means the direct method for solving diagonal blocks is out of memory)

Numerical experiments: 3D footing problem

Iteration counts for the block preconditioners
with varying physical parameters, K and ν

	$\nu = 0.2$ and varying K					
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$\mathcal{B}_D$	28	28	49	68	42	35
$\mathcal{B}_L$	20	20	27	37	26	24
$\mathcal{B}_U$	18	18	26	35	21	14
$\widehat{\mathcal{B}_D}$	28	28	49	68	42	42
$\widehat{\mathcal{B}_L}$	20	20	28	37	27	25
$\widehat{\mathcal{B}_U}$	21	21	27	35	22	24

	$K = 10^{-6}$ and varying ν					
	0.1	0.2	0.4	0.45	0.49	0.499
$\mathcal{B}_D$	72	68	51	46	35	26
$\mathcal{B}_L$	41	37	25	21	17	20
$\mathcal{B}_U$	38	35	25	21	17	20
$\widehat{\mathcal{B}_D}$	72	68	51	46	35	26
$\widehat{\mathcal{B}_L}$	41	37	25	21	17	20
$\widehat{\mathcal{B}_U}$	38	35	25	21	17	21

Cheaper strategy?

Block preconditioners robust w.r.t. discretization and physical parameters for the full stabilized P1-RT0-P0 system

Cheaper strategy?

Block preconditioners robust w.r.t. discretization and physical parameters for the full stabilized P1-RT0-P0 system

We have the following block form of the discrete problem:

$$\boxed{\mathcal{A}\begin{pmatrix} \boldsymbol{U}_b \\ \boldsymbol{U}_l \\ \boldsymbol{P} \\ \boldsymbol{W} \end{pmatrix} = \boldsymbol{b}, \quad \text{with} \quad \mathcal{A} = \begin{pmatrix} A_{bb} & A_{bl} & \alpha B_b^T & 0 \\ A_{bl}^T & A_{ll} & \alpha B_l^T & 0 \\ -\alpha B_b & -\alpha B_l & \frac{1}{M} M_p & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} \end{pmatrix}}$$

$$a(\boldsymbol{u}_h^b, \boldsymbol{v}_h^b) \rightarrow A_{bb}, \quad a(\boldsymbol{u}_h^l, \boldsymbol{v}_h^b) \rightarrow A_{bl}, \quad a(\boldsymbol{u}_h^l, \boldsymbol{v}_h^l) \rightarrow A_{ll},$$

$$-(p_h, \operatorname{div} \boldsymbol{v}_h^b) \rightarrow B_b, \quad -(p_h, \operatorname{div} \boldsymbol{v}_h^l) \rightarrow B_l, \quad -(p_h, \operatorname{div} \boldsymbol{r}_h) \rightarrow B_{\boldsymbol{w}},$$

$$(K^{-1}\mu_f \boldsymbol{w}_h, \boldsymbol{r}_h) \rightarrow M_w, \quad (p_h, q_h) \rightarrow M_p,$$

Perturbation of the bilinear form. Elimination of bubbles

- For the restriction of $a(\cdot,\cdot)$ onto the space spanned by bubble functions $\boldsymbol{V}_b$, we have

$$a_b(\boldsymbol{u}_b,\boldsymbol{v}_b) := a(\boldsymbol{u}_b,\boldsymbol{v}_b) = \sum_{T\in\mathcal{T}_h} a_{b,T}(\boldsymbol{u}_b,\boldsymbol{v}_b) = \sum_{T\in\mathcal{T}_h}\sum_{e,e'\in\partial T} u_e v_{e'}\, a_T(\boldsymbol{\Phi}_{e'},\boldsymbol{\Phi}_e).$$

- On each element, $T\in\mathcal{T}_h$, then introduce

$$d_{b,T}(\boldsymbol{u},\boldsymbol{v}) = (d+1)\sum_{e\in\partial T} u_e v_e a_T(\boldsymbol{\Phi}_e,\boldsymbol{\Phi}_e), \quad d_b(\boldsymbol{u},\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h} d_{b,T}(\boldsymbol{u},\boldsymbol{v}).$$

- Replacing $a_b(\cdot,\cdot)$ with $d_b(\cdot,\cdot)$ gives a perturbation, $a^D(\cdot,\cdot)$, of $a(\cdot,\cdot)$:

$$a^D(\boldsymbol{u},\boldsymbol{v}) := d_b(\boldsymbol{u}_b,\boldsymbol{v}_b) + a(\boldsymbol{u}_b,\boldsymbol{v}_l) + a(\boldsymbol{u}_l,\boldsymbol{v}_b) + a(\boldsymbol{u}_l,\boldsymbol{v}_l)$$

Lemma: A spectral equivalence result

The following inequalities hold:

$$a(\boldsymbol{u},\boldsymbol{u}) \le a^D(\boldsymbol{u},\boldsymbol{u}) \le \eta a(\boldsymbol{u},\boldsymbol{u}), \quad \text{for all} \quad \boldsymbol{u}\in\boldsymbol{V}_h,$$

where η depends on the shape regularity of the mesh.

Perturbation of the bilinear form. Elimination of bubbles

We define the following block form of the discrete operator:

$$\mathcal{A}^D \begin{pmatrix} \boldsymbol{U}_b \\ \boldsymbol{U}_l \\ \boldsymbol{P} \\ \boldsymbol{W} \end{pmatrix} = \boldsymbol{b}, \quad \text{with} \quad \mathcal{A}^D = \begin{pmatrix} D_{bb} & A_{bl} & \alpha B_b^T & 0 \\ A_{bl}^T & A_{ll} & \alpha B_l^T & 0 \\ -\alpha B_b & -\alpha B_l & \frac{1}{M} M_p & -\tau B_{\boldsymbol{w}} \\ 0 & 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} \end{pmatrix}$$

After eliminating the degrees of freedom corresponding to the bubble functions, we obtain:

$$\mathcal{A}^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & \alpha B_l^T - \alpha A_{bl}^T D_{bb}^{-1} B_b^T & 0 \\ -\alpha B_l + \alpha B_b D_{bb}^{-1} A_{bl} & \frac{1}{M} M_p + \alpha^2 B_b D_{bb}^{-1} B_b^T & -\tau B_{\boldsymbol{w}} \\ 0 & \tau B_{\boldsymbol{w}}^T & \tau M_w \end{pmatrix}.$$

We have the same degrees of freedom as in the original P1-RT0-P0 method for the three-field formulation.

Well-posedness for the bubble-eliminated system

Here, $\boldsymbol{X}_h^E$ denotes the discretized finite-element space after bubble elimination.

Theorem (Adler, Gaspar, Hu, Ohm, Rodrigo, & Zikatanov 2018)

If the full system is well-posed as shown before, then the bubble-eliminated system satisfies the following inequalities for $\boldsymbol{x}^E = (\boldsymbol{u}_l, p_h, \boldsymbol{w}_h)^T \in \boldsymbol{X}_h^E$ and $\boldsymbol{y}^E = (\boldsymbol{v}_l, q_h, \boldsymbol{r}_h)^T \in \boldsymbol{X}_h^E$,

$$\inf_{0 \neq \boldsymbol{x}^E \in \boldsymbol{X}_h^E} \sup_{0 \neq \boldsymbol{y}^E \in \boldsymbol{X}_h^E} \frac{(\mathcal{A}^E \boldsymbol{x}^E, \boldsymbol{y}^E)}{\|\boldsymbol{x}^E\|_{\mathcal{D}^E} \|\boldsymbol{y}^E\|_{\mathcal{D}^E}} \geq \gamma^*,$$

$$\sup_{0 \neq \boldsymbol{x}^E \in \boldsymbol{X}_h^E} \sup_{0 \neq \boldsymbol{y}^E \in \boldsymbol{X}_h^E} \frac{(\mathcal{A}^E \boldsymbol{x}^E, \boldsymbol{y}^E)}{\|\boldsymbol{x}^E\|_{\mathcal{D}^E} \|\boldsymbol{y}^E\|_{\mathcal{D}^E}} \leq \varsigma,$$

where,

$$\mathcal{D}^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & 0 & 0 \\ 0 & \alpha^2 B_b D_{bb}^{-1} B_b^T + c_p^{-1} M_p & 0 \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \tau^2 c_p A_{\boldsymbol{w}} \end{pmatrix},$$

with $\|\boldsymbol{x}^E\|_{\mathcal{D}^E}^2 = (\mathcal{D}^E \boldsymbol{x}^E, \boldsymbol{x}^E)$.
Thus, the bubble-eliminated system is well-posed w.r.t. the weighted norm defined above, independently of the discretization and physical parameters.

Norm-equivalent Preconditioners for the bubble-eliminated system

Block diagonal preconditioners:

$$\mathcal{B}_D^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & 0 & 0 \\ 0 & c_p^{-1} M_p + \alpha^2 B_b D_{bb}^{-1} B_b^T & 0 \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \tau^2 c_p A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

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where $c_p := \left(\frac{\alpha^2}{\lambda + \frac{2\mu}{d}} + \frac{1}{M} \right)^{-1}$ and

$$\widehat{\mathcal{B}_D^E} = \begin{pmatrix} Q_{\boldsymbol{u}}^E & 0 & 0 \\ 0 & Q_p^E & 0 \\ 0 & 0 & Q_{\boldsymbol{w}}^E \end{pmatrix}$$

where $Q_{\boldsymbol{u}}^E$: MG, Q_p^E: MG, and $Q_{\boldsymbol{w}}^E$: Hiptmair-Xu/ADS

Norm-equivalent Preconditioners for the bubble-eliminated system

Block diagonal preconditioners:

$$\mathcal{B}_D^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & 0 & 0 \\ 0 & c_p^{-1} M_p + \alpha^2 B_b D_{bb}^{-1} B_b^T & 0 \\ 0 & 0 & \tau M_{\boldsymbol{w}} + \tau^2 c_p A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

where $c_p := \left(\frac{\alpha^2}{\lambda + \frac{2\mu}{d}} + \frac{1}{M} \right)^{-1}$ and

$$\widehat{\mathcal{B}_D^E} = \begin{pmatrix} Q_{\boldsymbol{u}}^E & 0 & 0 \\ 0 & Q_p^E & 0 \\ 0 & 0 & Q_{\boldsymbol{w}}^E \end{pmatrix}$$

where $Q_{\boldsymbol{u}}^E$: MG, Q_p^E: MG, and $Q_{\boldsymbol{w}}^E$: Hiptmair-Xu/ADS

Theorem (Adler, Gaspar, Hu, O., Rodrigo, & Zikatanov 2019)

If the linear system is well-posed w.r.t $|||\cdot|||_D$, then we have

$$\kappa(\mathcal{B}_D^E \mathcal{A}) \le C_1 \text{ and } \kappa(\widehat{\mathcal{B}_D^E} \mathcal{A}) \le C_2$$

where C_1 and C_2 are constants independent of the physical parameters (λ, μ, M, K, α) and the discretization parameters (h, τ).

FoV-equivalent Preconditioners for the bubble-eliminated system: Left Preconditioning

Block lower triangular preconditioner:

$$\mathcal{B}_L^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & 0 & 0 \\ -\alpha B_l + \alpha B_b D_{bb}^{-1} A_{bl} & c_p^{-1} M_p + \alpha^2 B_b D_{bb}^{-1} B_b^T & 0 \\ 0 & \tau B_{\mathbf{w}}^T & \tau M_{\mathbf{w}} + c_p \tau^2 A_{\mathbf{w}} \end{pmatrix}^{-1}$$

FoV-equivalent Preconditioners for the bubble-eliminated system: Left Preconditioning

Block lower triangular preconditioner:

$$\mathcal{B}_L^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & 0 & 0 \\ -\alpha B_l + \alpha B_b D_{bb}^{-1} A_{bl} & c_p^{-1} M_p + \alpha^2 B_b D_{bb}^{-1} B_b^T & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} + c_p \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

Inexact block lower triangular preconditioner:

$$\widehat{\mathcal{B}_L^E} = \begin{pmatrix} Q_{\boldsymbol{u}}^{E\,-1} & 0 & 0 \\ -\alpha B_l + \alpha B_b D_{bb}^{-1} A_{bl} & Q_p^{E\,-1} & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & Q_{\boldsymbol{w}}^{E\,-1} \end{pmatrix}^{-1}$$

FoV-equivalent Preconditioners for the bubble-eliminated system: Left Preconditioning

Block lower triangular preconditioner:

$$\mathcal{B}_L^E = \begin{pmatrix} A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl} & 0 & 0 \\ -\alpha B_l + \alpha B_b D_{bb}^{-1} A_{bl} & c_p^{-1} M_p + \alpha^2 B_b D_{bb}^{-1} B_b^T & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & \tau M_{\boldsymbol{w}} + c_p \tau^2 A_{\boldsymbol{w}} \end{pmatrix}^{-1}$$

Inexact block lower triangular preconditioner:

$$\widehat{\mathcal{B}_L^E} = \begin{pmatrix} Q_{\boldsymbol{u}}^{E\,-1} & 0 & 0 \\ -\alpha B_l + \alpha B_b D_{bb}^{-1} A_{bl} & Q_p^{E\,-1} & 0 \\ 0 & \tau B_{\boldsymbol{w}}^T & Q_{\boldsymbol{w}}^{E\,-1} \end{pmatrix}^{-1}$$

Theorem (Adler, Gaspar, Hu, O., Rodrigo, & Zikatanov 2019)

If the linear system is well-posed w.r.t $|||\cdot|||_D$, then we have

- $\mathcal{B}_L^E$ *and* $\mathcal{A}^E$ *are FoV-equivalent*
- $\widehat{\mathcal{B}_L^E}$ *and* $\mathcal{A}^E$ *are FoV-equivalent if* $\|I_{\boldsymbol{u}} - Q_{\boldsymbol{u}}^E(A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl})\|_{(A_{ll} - A_{bl}^T D_{bb}^{-1} A_{bl})} \leq \delta < 1$
- *FoV-equivalent constants are independent of the physical parameters* $(\lambda, \mu, M, K, \alpha)$ *and the discretization parameters* (h, τ)
- *Preconditioned GMRES converges uniformly*

Iteration counts for the block preconditioners
with varying discretization parameters

$\tau \backslash h$	$\mathcal{B}_D^E$					$\mathcal{B}_L^E$					$\mathcal{B}_U^E$				
	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
0.1	25	31	36	39	39	18	20	21	20	19	17	20	21	21	20
0.01	27	26	25	30	34	14	13	17	19	19	13	14	17	18	19
0.001	27	28	27	22	25	13	13	12	13	16	9	11	11	13	15
0.0001	22	25	25	24	22	10	11	11	11	11	9	9	9	10	11

$\tau \backslash h$	$\widehat{\mathcal{B}_D^E}$					$\widehat{\mathcal{B}_L^E}$					$\widehat{\mathcal{B}_U^E}$				
	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
0.1	25	31	36	39	39	18	20	21	20	19	17	20	21	21	20
0.01	27	26	25	30	34	14	14	17	19	19	13	14	17	18	19
0.001	27	28	27	22	25	13	14	12	13	16	9	11	12	14	15
0.0001	22	25	25	24	22	10	11	11	12	11	9	9	9	10	11

Numerical experiments: Mandel problem

Iteration counts for the block preconditioners with varying physical parameters K and ν.

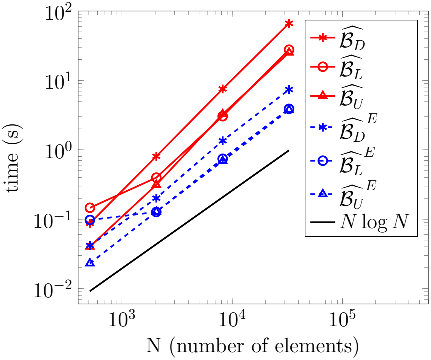
	$\nu = 0.0$ and varying K					
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$\mathcal{B}_D^E$	36	38	42	34	23	19
$\mathcal{B}_L^E$	14	15	19	19	11	7
$\mathcal{B}_U^E$	23	22	21	19	11	3
$\widehat{\mathcal{B}}_D^E$	49	38	42	34	23	14
$\widehat{\mathcal{B}}_L^E$	17	18	19	19	11	8
$\widehat{\mathcal{B}}_U^E$	34	24	21	19	11	2

	$K = 10^{-6}$ and varying ν					
	0.1	0.2	0.4	0.45	0.49	0.499
$\mathcal{B}_D^E$	43	50	43	43	43	30
$\mathcal{B}_L^E$	20	23	17	15	13	8
$\mathcal{B}_U^E$	24	28	23	23	21	16
$\widehat{\mathcal{B}}_D^E$	43	50	43	43	43	24
$\widehat{\mathcal{B}}_L^E$	20	23	18	17	14	12
$\widehat{\mathcal{B}}_U^E$	25	29	24	24	26	14

Numerical experiments: Mandel problem

Performance comparison between the block diagonal, block upper triangular and block lower triangular preconditioners for the full and the bubble-eliminated systems

Â (timing results versus mesh size)



Solving the bubble-eliminated system is faster than solving the full-bubble system

Numerical experiments: 3D footing problem

Iteration counts for the block preconditioners
with varying discretization parameters

τ \ h	$\mathcal{B}_D^E$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\mathcal{B}_L^E$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\mathcal{B}_U^E$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
0.1	61	65	66	$*$	41	41	39	$*$	39	39	38	$*$
0.01	54	58	66	$*$	39	42	43	$*$	33	39	41	$*$
0.001	58	58	53	$*$	37	39	40	$*$	28	32	35	$*$
0.0001	59	61	60	$*$	35	38	38	$*$	29	29	30	$*$

τ \ h	$\widehat{\mathcal{B}_D^E}$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\widehat{\mathcal{B}_L^E}$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\widehat{\mathcal{B}_U^E}$ $\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
0.1	61	65	66	66	41	41	39	39	40	40	38	37
0.01	54	58	66	70	39	42	43	43	33	39	41	42
0.001	58	58	53	61	37	39	40	43	28	32	35	40
0.0001	58	61	60	55	35	38	38	38	29	30	30	32

($*$ means the direct method for solving diagonal blocks is out of memory)

Numerical experiments: 3D footing problem

Iteration counts for the block preconditioners
with varying physical parameters, K and ν

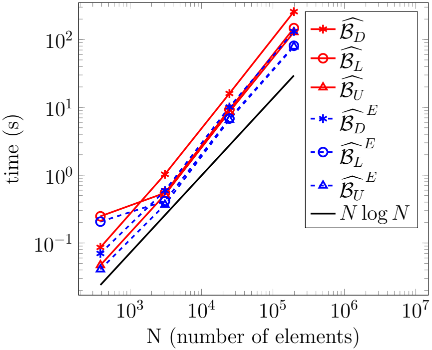
	$\nu = 0.2$ and varying K					
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}
$\mathcal{B}_D^E$	33	33	51	66	60	61
$\mathcal{B}_L^E$	20	20	29	43	38	35
$\mathcal{B}_U^E$	20	20	29	41	28	18
$\widehat{\mathcal{B}_D^E}$	33	33	51	66	60	61
$\widehat{\mathcal{B}_L^E}$	22	22	30	43	38	36
$\widehat{\mathcal{B}_U^E}$	22	22	29	41	29	29

	$K = 10^{-6}$ and varying ν					
	0.1	0.2	0.4	0.45	0.49	0.499
$\mathcal{B}_D^E$	70	66	53	48	43	28
$\mathcal{B}_L^E$	46	43	32	28	24	21
$\mathcal{B}_U^E$	44	41	31	27	24	21
$\widehat{\mathcal{B}_D^E}$	70	66	53	48	43	28
$\widehat{\mathcal{B}_L^E}$	46	43	32	28	24	22
$\widehat{\mathcal{B}_U^E}$	44	41	31	28	24	22

Numerical experiments: 3D footing problem

Performance comparison between the block diagonal, block upper triangular and block lower triangular preconditioners for the full and the bubble-eliminated systems

Â (timing results versus mesh size)



Conclusions

1. Design of parameter-robust preconditioners:
   - Norm-equivalent preconditioners
   - Field-of-values preconditioners

   based on a stabilization of the $P1 - RT0 - P0$ finite-element discretization for a three-field formulation of the poroelasticity system.

2. Same idea for the design of preconditioners robust w.r.t. discretization and physical parameters for the corresponding bubble-eliminated approach which has the same number of unknowns as in the initial P1-RT0-P0 discretization.

Conclusions

1. Design of parameter-robust preconditioners:
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   based on a stabilization of the $P1 - RT0 - P0$ finite-element discretization for a three-field formulation of the poroelasticity system.

2. Same idea for the design of preconditioners robust w.r.t. discretization and physical parameters for the corresponding bubble-eliminated approach which has the same number of unknowns as in the initial P1-RT0-P0 discretization.

Thank you for your attention!