Geometric errors in surface finite element methods

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parts joint work with

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Outline

1. Canonical surface FEM
    - Implicit surface representations
    - FEM
    - Consistency errors and a priori estimates
2. Eigenvalue problems
    - Definitions
    - Results of canonical arguments
    - Quadrature-based superconvergence argument
3. A posteriori estimates
    - Estimates assuming canonical implicit representation
    - Parametric representation of surfaces
    - New estimates
4. Conclusion

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Prologue: Basic FEM terminology

Poisson model problem: Solve

$$-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega.$$

Sobolev space, forms, and norm: Let

$$a(u, v) := \int_\Omega \nabla u \nabla v, \quad m(f, v) := \int_\Omega f v.$$

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L_2(\Omega)} = a(u, u)^{1/2},$$

$$H_0^1(\Omega) = \{u \ s.t. \ \|u\|_{H_0^1(\Omega)} < \infty, \quad u = 0 \text{ on } \partial\Omega.\}$$

Infinite dimensional vector space.

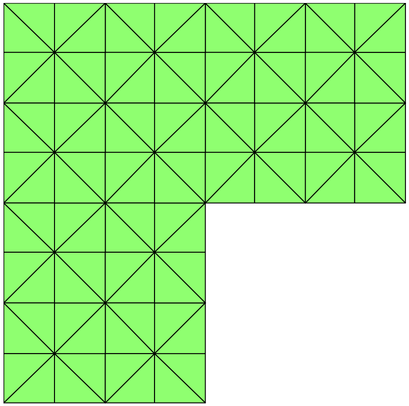
Weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) = m(f, v), \quad \text{all } v \in H_0^1(\Omega).$$

Find basis for $H_0^1(\Omega) \rightsquigarrow$ infinite-dimensional set of linear equations.

Euclidean FEM

Mesh: $\mathcal{T}_h$ is a decomposition of Ω into triangles of diameter h.



Euclidean FEM

Mesh: $\mathcal{T}_h$ is a decomposition of Ω into triangles of diameter h.

Finite element subspace: Elements of $S_h \subset H_0^1(\Omega)$ are

- Continuous
- 0 on $\partial\Omega$
- polynomials of degree r over each $T \in \mathcal{T}_h$.

S_h is a *(finite dimensional vector space.*

Galerkin's method: Find $u_h \in S_h$ s.t.

$$a(u_h, v_h) = m(f, v_h), \quad v_h \in S_h.$$

Basis for $S_h \rightsquigarrow$ finite dimensional set of linear equations.

Projection property: u_h is the orthogonal projection of u onto S_h w.r.t. a:

$$\|u - u_h\|_{H_0^1(\Omega)} = \inf_{\chi \in S_h} \|u - \chi\|_{H_0^1(\Omega)}.$$

Basic error estimates

If u is smooth enough,

$$\|u - u_h\|_{H_0^1(\Omega)} \leq \inf_{\chi \in S_h} \|u - \chi\|_{H_0^1(\Omega)} \leq C h^r,$$

$$\|u - u_h\|_{L_2(\Omega)} \leq C h^{r+1}.$$

Relates error to cost and properties of method: h, r.

Variational crimes/Consistency errors

Variational crimes/Consistency errors: Sometimes we define u_h via forms "close to" a and m:

$$A(u_h, v_h) = M(u_h, v_h), \quad \text{all } v_h \in S_h,$$

with $A \approx a$ and $M \approx m$.

Must account for loss of projection property in error analysis:

$$\|u - u_h\|_{H^1(\Omega)} \le \inf_{\chi \in S_h} \|u - \chi\|_{H^1(\Omega)} + \|A - a\|_* + \|M - m\|_*.$$

(Effects on convergence rate depend on the situation...).

1. Laplace-Beltrami problem

Definitions:

- γ is a compact, 2-dimensional C^2 surface without boundary in $\mathbb{R}^3$.
- f is (given) data satisfying $\int_\gamma f \,\mathrm{d}\sigma = 0$.
- ∇_γ, Δ_γ are the tangential gradient and Laplace-Beltrami operator.

Model problem (strong form):

$$-\Delta_\gamma u = f \text{ on } \gamma.$$

Dirichlet form and L_2 inner product:

$$a(u,v) := \int_\gamma \nabla_\gamma u \nabla_\gamma v \,\mathrm{d}\sigma, \quad m(f,v) := \int_\gamma f v \,\mathrm{d}\sigma.$$

Weak form of the Laplace-Beltrami problem: Find $u \in H^1(\gamma)$ s.t.

$$a(u,v) = m(f,v) \text{ for all } v \in H^1(\gamma).$$

We require $\int_\gamma u \,\mathrm{d}\sigma = 0$ to ensure uniqueness.

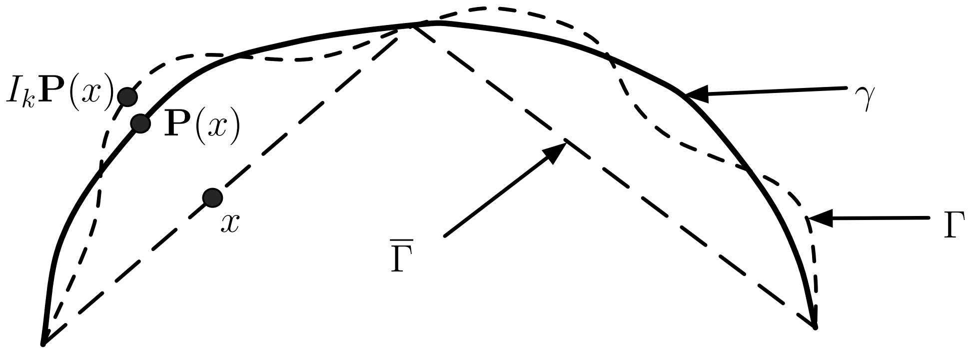
Applications of surface PDE

Why solve the Laplace-Beltrami problem?

1. Geometry: Mean curvature flow, etc.
2. Image and surface processing
3. Physical modeling: Surface tension in two-phase flow; biomembranes
4. Shape registration: Spectrum can serve as a "shape DNA"

Surface FEM

- *Base discrete surface:* $\overline{\Gamma}$ is a polyhedron with triangular or quad faces.
- **Basic mapping assumption:** There is a “reasonably nice” map $\mathbf{P} : \overline{\Gamma} \to \gamma$.
- *Polynomial surface approximation:* $\Gamma = I_k \mathbf{P}(\overline{\Gamma})$ with I_k a degree-k Lagrange interpolant.



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- *Polynomial surface approximation:* $\Gamma = I_k \mathbf{P}(\overline{\Gamma})$ with I_k a degree-k Lagrange interpolant.
- *Meshes:* $\overline{\mathcal{T}}$ is the set of faces of $\overline{\Gamma}$, $\mathcal{T}$ is the faces of Γ.
- *Finite element space:* $S_{\mathcal{T}}$ is the piecewise degree-r polynomials over Γ.
- *Data:* f is defined on γ, so have to define data F on Γ.
- *Forms on* Γ*:*

$$A(U, V) := \int_\Gamma \nabla_\Gamma U \nabla_\Gamma V \, \mathrm{d}\sigma_{\mathcal{T}}, \quad M(F, V) := \int_\Gamma F V \, \mathrm{d}\sigma_{\mathcal{T}}.$$

- *Finite element method:* Find $U \in S_{\mathcal{T}}$ such that $\int_\Gamma U \, \mathrm{d}\sigma_{\mathcal{T}} = 0$ and

$$A(U, V) = M(F, V), \quad V \in S_{\mathcal{T}}.$$

Choosing P

Canonical choice historically for C^2 surfaces: Implicit representation.

Viewpoint: $\gamma = \{x : d(x) = 0\}$ with d the signed distance function.

Then: For x lying in a sufficiently small tubular neighborhood U of γ,

- *Orthogonal closest-point projection onto γ:*

$$\mathbf{P}_d(x) := x - d(x)\vec{\nu}(x)$$

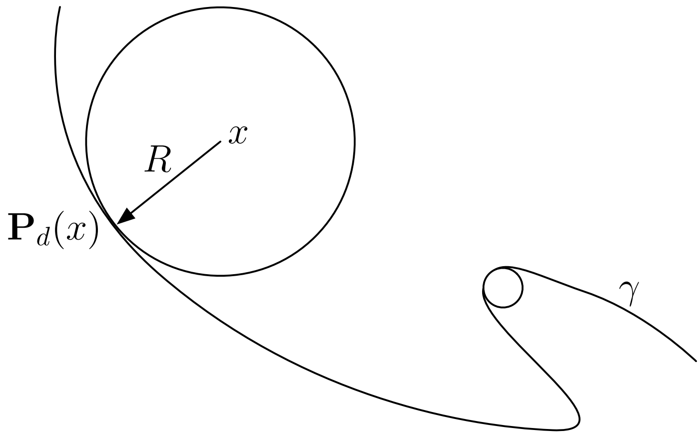
  with $\nu = \nabla d$ the unit normal on γ.

Ups and downs:

+ Correct theoretical properties in FEM.
- Often difficult to access in codes (explicit formulas only for sphere, torus).
- Surface regularity less than C^2?

We'll look at other options later...

Curvature and the closest point projection



Notes:

- *Curvature:* $\kappa(\mathbf{P}_d(x)) = 1/R$ with R the maximum radius of open balls tangent to but not intersecting γ.
- *Closest point projection:* Uniquely defined on a tubular neighborhood of γ having width $\inf_{x\in\gamma} \frac{1}{|\kappa(x)|}$.

Surface regularity and the distance function

Two distinct regimes of surface regularity:

1. γ is $C^{1,1}$ or smoother (can be locally described via a $C^{1,1}$ diffeomorphism):
   - The distance function, closest point projection behave as described above.
   - Distance function inherits surface regularity: γ is $C^k \Rightarrow d$ is also C^k.
2. For any γ not $C^{1,1}$ (say, $C^{1,\alpha}$ with $\alpha < 1$):
   - d does NOT inherit surface regularity: d is only Lipschitz.
   - $\mathbf{P}_d$ is not uniquely defined on ANY open neighborhood of γ.
   - Established in [Lucas, 1957] and [Federer, 1959].

Geometric consistency error

Dirichlet consistency matrix: With $\mathbf{E}_{\mathbf{P}_d}$ a matrix determined by using change of variables formulas for the mapping $\mathbf{P}_d$ and $U^\ell = U \circ \mathbf{P}_d^{-1}$,

$$A(U,V) - a(U^\ell, V^\ell) = \int_\gamma \mathbf{E}_{\mathbf{P}_d} \nabla_\gamma U^\ell \nabla_\gamma V^\ell \, \mathrm{d}\sigma.$$

- Computing $\mathbf{E}_{\mathbf{P}_d}$ requires computing distance function d and derivatives.
- *Order of consistency error:* On a triangle T of size h,

  $$\|\mathbf{E}_{\mathbf{P}_d}\|_{L_\infty(T)} \lesssim \|d\|_{L_\infty(T)} + \|\vec{\nu} - \vec{\nu}_h\|^2_{L_\infty(T)} \lesssim h^{k+1} + h^{2k} \lesssim h^{k+1},$$

  where $\vec{\nu}$ and $\vec{\nu}_h$ are normals to γ and Γ.
- An $O(h^{k+1})$ consistency error is observed *essentially independently of the method used to construct* Γ (interpolation of $\mathbf{P}_d$ isn't necessary in practice!).

A priori estimates for surface FEM

Theorem 1 (Dz88, De09)**.** *For discrete data F consistently chosen,*

$$\begin{aligned}
\|\nabla_\gamma(u - U^\ell)\|_{L_2(\gamma)} &\lesssim h^r \|u\|_{H^{r+1}(\gamma)} + \|\mathbf{E}_{\mathbf{P}_d}\|_{L_\infty(\gamma)} \|\nabla_\gamma u\|_{L_2(\gamma)} \\
&\lesssim h^r \|u\|_{H^{r+1}(\gamma)} + h^{k+1} \|\nabla_\gamma u\|_{L_2(\gamma)}, \\
\|u - U_T^\ell - \frac{1}{|\gamma|} \int_\gamma (u - U_T^\ell)\|_{L_2(\gamma)} &\lesssim h^{r+1} \|u\|_{H^{r+1}(\gamma)} + \|\mathbf{E}_{\mathbf{P}_d}\|_{L_\infty(\gamma)} \|\nabla_\gamma u\|_{L_2(\gamma)} \\
&\lesssim h^{r+1} \|u\|_{H^{r+1}(\gamma)} + h^{k+1} \|\nabla_\gamma u\|_{L_2(\gamma)}.
\end{aligned}$$

Notes:

- Error consists of a Galerkin error and a geometric consistency error.
- Geometric error is the same for energy and L_2 norms.
- $r = k = 1$: Previous estimates require C^3 regularity. This is too much since $C^2 \Rightarrow u \in H^2(\gamma)$. Requirement is reduced to C^2 in recent joint work w/Bonito and Nochetto.

A (too?) general statement

Metatheorem: Geometric consistency errors are of order $k+1$ for any quantity of interest (various norms, point values...) and any standard surface FEM (mixed, DG, HDG, cut/trace, FEEC, parabolic problems...) for elliptic problems on surfaces.

Proof: See lots of FEM literature starting with [Dziuk 88] (also BEM literature starting with [Nedelec '78], [Bendali '84]...)

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Theme of this talk: Things aren't always that simple!

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Laplace-Beltrami Eigenvalue Problem

- **Strong form:** Find (u, λ) such that:

$$-\Delta_\gamma u = \lambda u,$$

- **Weak eigenvalue problem:** Find $(u, \lambda) \in H^1(\gamma)/\mathbb{R} \times \mathbb{R}^+$ such that

$$a(u, v) = \lambda m(u, v) \qquad \forall v \in H^1(\gamma).$$

- **Finite element approximation:** Find $(U, \Lambda) \in S_{\mathcal{T}}/\mathbb{R} \times \mathbb{R}^+$ such that

$$A(U, V) = \Lambda M(U, V), \quad V \in S_{\mathcal{T}}.$$

- **Eigenfunction bound:** Let $\boldsymbol{P}_\lambda$ be the $L_2(\gamma)$ projection onto eigenspace associated with λ. For an SFEM eigenpair (U, Λ) associated to an eigenvalue λ of $-\Delta_\gamma$, we have

$$\|U - \boldsymbol{P}_\lambda U\|_{H^1(\gamma)} \lesssim h^r + h^{k+1},$$
$$\|U - \boldsymbol{P}_\lambda U\|_{L_2(\gamma)} \lesssim h^{r+1} + h^{k+1}.$$

Eigenvalue Errors

Theorem 2 (Eigenvalue Bound)**.** *Let λ be an eigenvalue of the surface eigenvalue problem and let (U,Λ) be a surface FEM eigenpair associated with λ. Then*

$$|\lambda - \Lambda| \leq \underbrace{\|U - \boldsymbol{P}_\lambda U\|^2_{H^1(\gamma)}}_{O(h^{2r})+O(h^{2k+2})} + \lambda \underbrace{\|U - \boldsymbol{P}_\lambda U\|^2_{L_2(\gamma)}}_{O(h^{2r+2})+O(h^{2k+2})}$$

$$+\Lambda \underbrace{|m(U,U) - M(U,U)|}_{Geometric} + \underbrace{|a(U,U) - A(U,U)|}_{Geometric}.$$

Obvious eigenvalue error bound:

$$|\lambda - \Lambda| \lesssim h^{2r} + \|\mathbf{E}_{\mathbf{P}_d}\|_{L_\infty} \lesssim h^{2r} + h^{k+1}.$$

Some Test Shapes

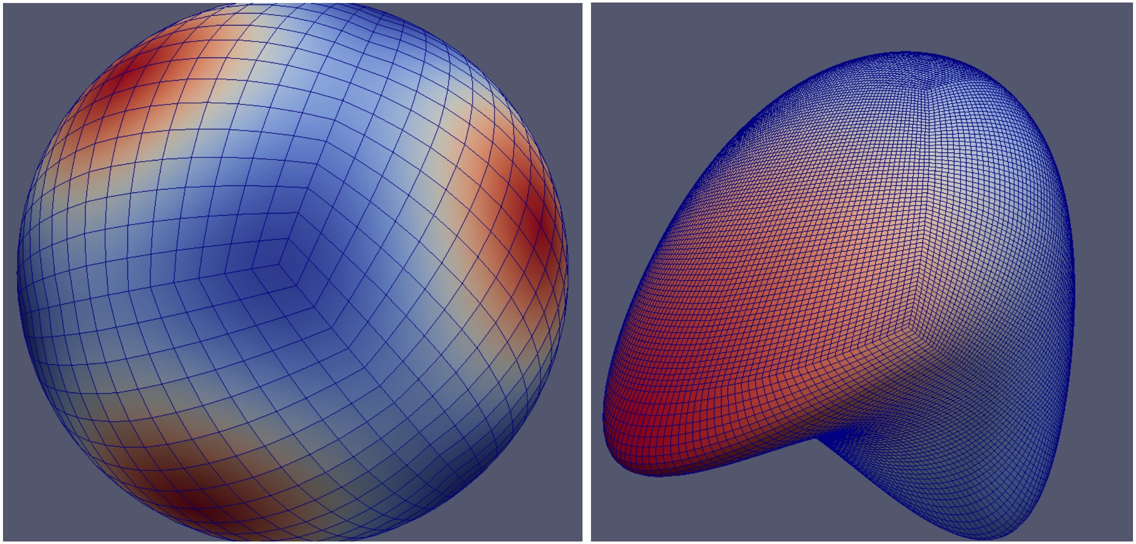
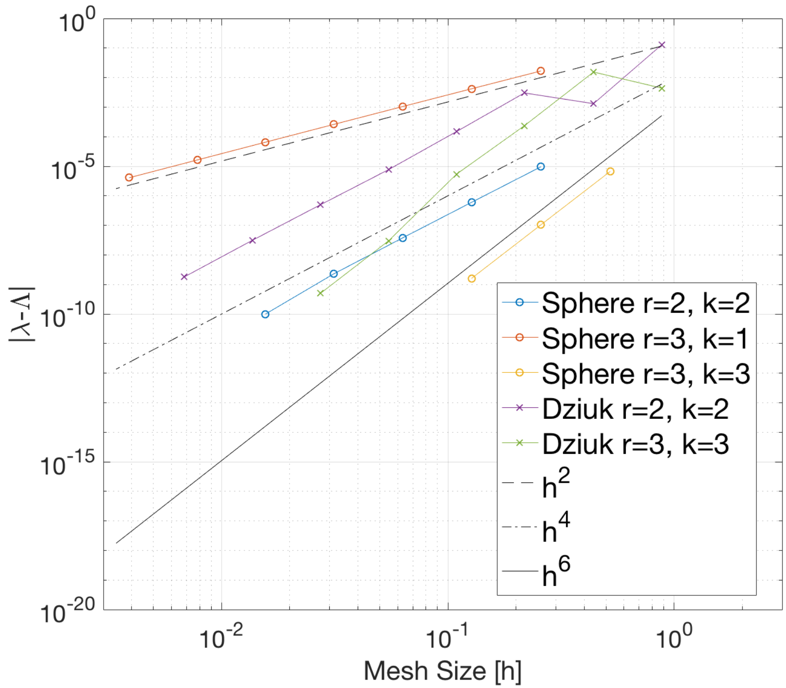


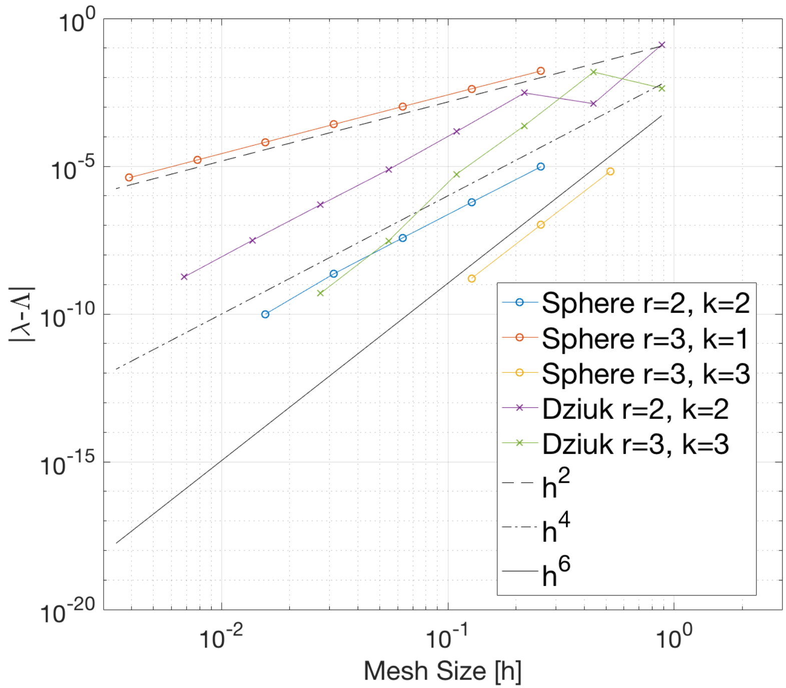
Figure 1: Sphere and Dziuk surface used in deal.ii computations of eigenvalues.

Numerical Experiments: Quadrilateral Elements



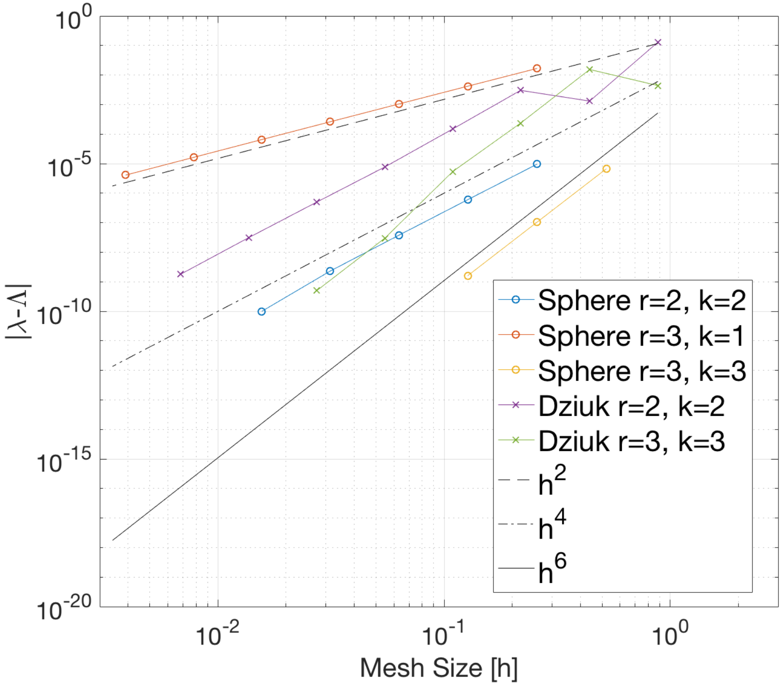
Looking for : $O(h^{k+1})$.

Numerical Experiments: Quadrilateral Elements



Strange Behavior: Geometric error is $O(h^{2k})$ rather than the expected $O(h^{k+1})$.

Numerical Experiments: Quadrilateral Elements



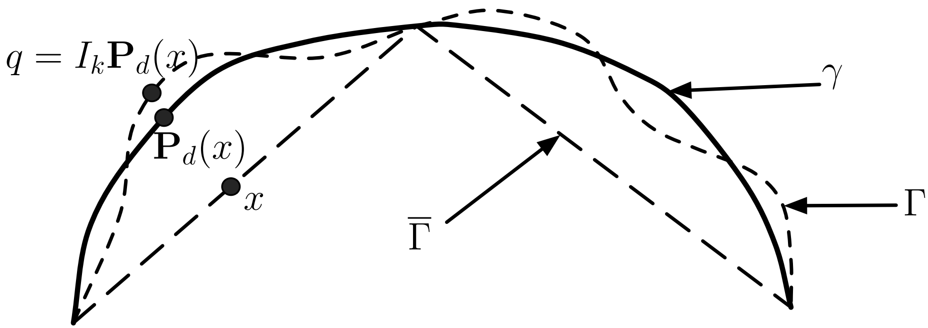
Important observation: deal.ii constructs Γ using interpolation of $\mathbf{P}_d$ at Gauss-Lobatto interpolation points, NOT canonical Lagrange points.

An Explanation of Superconvergence

Lemma 1. *Up to terms of order* h^{2k},

$$|m(V,V) - M(V,V)| \le \left| \int_\Gamma V(q)^2 d(q) \sum_{i=1}^{n} \frac{\kappa_i(\mathbf{P}_d(q))}{1 + d(q)\kappa_i(\mathbf{P}_d(q))}\, d\Sigma \right|,$$

where $\{\kappa_i\}_{i=1}^n$ *are the principal curvatures of the surface.*



Geometric Error Acts Like Quadrature Error

- **Exploit distance function:** The zeros of $d(q)$, $\{q_j\}_{j=1}^{N}$, on each face of Γ are the interpolation points used to create Γ.

- **Create quadrature rule:** Use the zeros of $d(q)$ to create a quadrature rule:

$$QUAD := \sum_{T\subset\Gamma}\sum_{j=1}^{N} W_j V(q_j)^2 \cancelto{0}{d(q_j)} \sum_{i=1}^{n} \frac{\kappa_i(\mathbf{P}(q_j))}{1 + d(q_j)\kappa_i(\mathbf{P}(q_j))} = 0$$

Theorem 3 (Quadrature Error)**.** *Up to terms of order* h^{2k},

$$\begin{aligned}|m(V,V) - M(V,V)| &\leq \left| \int_\Gamma V(q)^2 d(q) \sum_{i=1}^{n} \frac{\kappa_i(\mathbf{P}(q))}{1 + d(q)\kappa_i(\mathbf{P}(q))}\, d\Sigma \right| \\ &= \left| \int_\Gamma V(q)^2 d(q) \sum_{i=1}^{n} \frac{\kappa_i(\mathbf{P}(q))}{1 + d(q)\kappa_i(\mathbf{P}(q))}\, d\Sigma - QUAD \right|.\end{aligned}$$

Conclusion for quad meshes

Corollary 4 (Superconvergence in deal.ii Computations)**.** *If* $degree - k$ *interpolation points based on Gauss-Lobatto quadrature are used in the construction of* Γ, U *is the SFEM eigenfunction of* Λ, *and* $\boldsymbol{P}_\lambda U$ *has enough regularity, then*

$$|m(U,U) - M(U,U)| \lesssim h^{2k},$$
$$|a(U,U) - A(U,U)| \lesssim h^{2k},$$

and

$$|\lambda - \Lambda| \lesssim h^{2r} + h^{2k}.$$

Note: Tensor product of $k+1$ points used in the 1D Gauss-Lobatto quadrature rule yields a quadrature rule exact for degree $2k-1$.

Triangular meshes

Notes:

- Interpolation points are standard Lagrange points.
- Elementwise quadrature error for associated quadrature rule is $O(h^{k+1})$.
- Computational observation: Expected order h^{k+1} for odd k, superconvergent order h^{k+2} for even k.
- Observed orders were robust: Only exception was nodes perturbed off of surface with bias in one direction (e.g., outside of surface).
- Could be explained within our framework by known superconvergence phenomena for semi-structured meshes such as ones in which adjacent triangles form near-parallelograms.
- We didn't seem to have such structured meshes, but did not explore further down the superconvergence rabbit hole.

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Setting

Back to the Laplace-Beltrami source problem:

$$-\Delta_\gamma u = f \text{ on } \gamma.$$

FEM: Find $U_{\mathcal{T}} \in S_{\mathcal{T}}$ s.t.

$$A(U_{\mathcal{T}}, V) = (F, V), \quad V \in S_{\mathcal{T}}.$$

Goal: A posteriori (computable) estimates that bound the error:

$$\|\nabla_\gamma (u - U_{\mathcal{T}})\|_{L_2(\gamma)} \leq \mathcal{F}(U_{\mathcal{T}}, F) + \mathcal{G}(U_{\mathcal{T}}, F),$$

where $\mathcal{F}$ is a computable term controlling the Galerkin error and $\mathcal{G}$ is a computable term controlling the geometric error.

Note: Computing $\mathcal{G}$ will require computing the map $\mathbf{P}$ between Γ and γ.
How does our choice of $\mathbf{P}$ *affect the estimates?*

A posteriori estimates on implicit surfaces

Fundamental assumption: γ is represented *in implementation* using the closest point projection $\mathbf{P_d}(x) = x - d(x)\vec{\nu}(x)$.

Residual indicator: For $T \in \mathcal{T}$,

$$\eta_T = h_T \|F + \Delta_\Gamma U_\mathcal{T}\|_{L_2(T)} + h_T^{1/2} \|[\![\nabla_\Gamma U_\mathcal{T}]\!]\|_{L_2(\partial T)}.$$

Theorem 5 (De-Dz '07). *Assume* $F(x) = J_{\mathbf{P}_d}(f \circ \mathbf{P}_d)$ *with* $J_{\mathbf{P}_d}$ *the Jacobian of* $\mathbf{P}_d$*. Then*

$$\|\nabla_\gamma (u - U_\mathcal{T}^\ell)\|_{L_2(\gamma)}^2 \lesssim \sum_{T \in \mathcal{T}} \eta_T^2 + \|\mathbf{E}_{\mathbf{P}_d}\|_{L_\infty(\Gamma)}^2 \|\nabla_\Gamma U_\mathcal{T}\|_{L_2(\Gamma)}^2.$$

Notes:

- Galerkin error+geometric consistency error
- Everything is computable IF we can compute d and its derivatives (needed to compute/estimate $\mathbf{E}_{\mathbf{P}_d}$).
- Can also work with a more general level set function, but still need to approximate d.

Summary: Estimates on implicit surfaces

Pluses:

+ **Geometric error is of order h^{k+1}: "Superconvergent".**

Minuses:

- Analytical framework requires C^2 surface.
- **A posteriori estimates require evaluation of distance function: Only explicitly available for sphere and torus!**

Second option for $\mathbf{P}$

Framework: There is an elementwise-smooth bi-Lipschitz map $\mathbf{P} : \overline{\Gamma} \to \gamma$ *which we have access to in our code.*

Simple example: γ is the graph of a function g over a Euclidean domain Ω; $\mathbf{P} \neq \mathbf{P}_d$ is the "vertical" map induced by g.

Advantages:

1. More flexibility in representing smooth surfaces
2. Allows for less then C^2 surfaces.

Drawback: Theoretical properties aren't so nice!

Consistency errors

Assume $\mathbf{P}$ is an arbitrary "reasonable" parametric map:

- Error representation: With $\mathbf{E}_{\boldsymbol{P}}$ a matrix derived from change of variables,

$$A(U,V) - a(U \circ \boldsymbol{P}^{-1}, V \circ \boldsymbol{P}^{-1}) = \int_\gamma \mathbf{E}_{\mathbf{P}} \nabla_\gamma (U \circ \boldsymbol{P}^{-1}) \nabla_\gamma (V \circ \boldsymbol{P}^{-1})\, \mathrm{d}\sigma.$$

- Computing $\mathbf{E}_{\mathbf{P}}$ only requires access to $\mathbf{P}$.
- Standard arguments for isoparametric FEM yield

$$\|\mathbf{E}_{\mathbf{P}}\|_{L_\infty(T)} \lesssim h^k.$$

The moral of the story: $O(h^{k+1})$ geometric errors are observed for smooth surfaces independent of $\mathbf{P}$ used in implementation. *Thus we should use $\mathbf{P}_d$ for theoretical purposes.*

A posteriori estimates: Parametric viewpoint

Theorem 6 (BCMMN, 2016)**.** *Let* $F = J_{\mathbf{P}}(f \circ \mathbf{P})$. *Then under reasonable assumptions,*

$$\|\nabla_\gamma(u - U_{\mathcal{T}} \circ \mathbf{P})\|^2_{L_2(\gamma)} \lesssim \sum_{T \in \mathcal{T}} \eta_T^2 + \|\nabla(\mathbf{P} - I_k \mathbf{P})\|^2_{L_\infty(\overline{\Gamma})}.$$

Properties:

+ Practical computation uses $\mathbf{P}$: Flexible!

+ Allows for less-than-C^2 surfaces.

+ AFEM convergence, optimality proved.

- Geometric consistency error $\|\nabla(\mathbf{P} - I_k \mathbf{P})\|_{L_\infty(\overline{\Gamma})}$ is only order h^k, *not order* h^{k+1} *as in the implicit formulation.*

- AFEM significantly overrefines to resolve geometric error.

A posteriori estimates: Merged perspective

Basic idea: Use generic $\mathbf{P}$ for *implementation*, but use $\mathbf{P}_d$ *for theory.*

The heart of our result:

$$\|\mathbf{E}_{\mathbf{P}_d}\|_{L_\infty(\gamma)} \lesssim \|\mathbf{P} - I_k\mathbf{P}\|_{L_\infty(\overline{\Gamma})} + \|\nabla(\mathbf{P} - I_k\mathbf{P})\|^2_{L_\infty(\overline{\Gamma})} =: \epsilon_{\mathcal{T}}.$$

Theorem 7 (De.-Bonito)**.** *Assume that γ is C^2, and that a parametric FEM is used with $F = J_{\mathbf{P}}(f \circ \mathbf{P})$. Then under reasonable assumptions,*

$$\|\nabla_\gamma(u - U_{\mathcal{T}})\|^2_{L_2(\gamma)} \lesssim \sum_{T \in \mathcal{T}} \eta_T^2 + \epsilon_{\mathcal{T}}^2.$$

Notes:

1. $\epsilon_{\mathcal{T}}$ is computable using only information from the parametric representation, but heuristically $\epsilon_{\mathcal{T}} \lesssim h^{k+1}$.
2. Central observation in proofs: $\mathbf{P}_d$ is the closest point projection implies

$$|x - \mathbf{P}_d(x)| \leq |x - \mathbf{P}(x)|.$$

Numerical experiments

Computational setup for Experiment 1:

- **Smooth geometry:** γ is a half-sphere (smooth) $\rightsquigarrow$ uniform geometric refinement.
- **Rough solution:**. u is singular at the north pole $\rightsquigarrow$ localized PDE refinement at pole.
- **Software:** Computations were performed using deal.ii.
- **Adaptive algorithm:** Selectively choose elements to subdivide based on elementwise quantities:

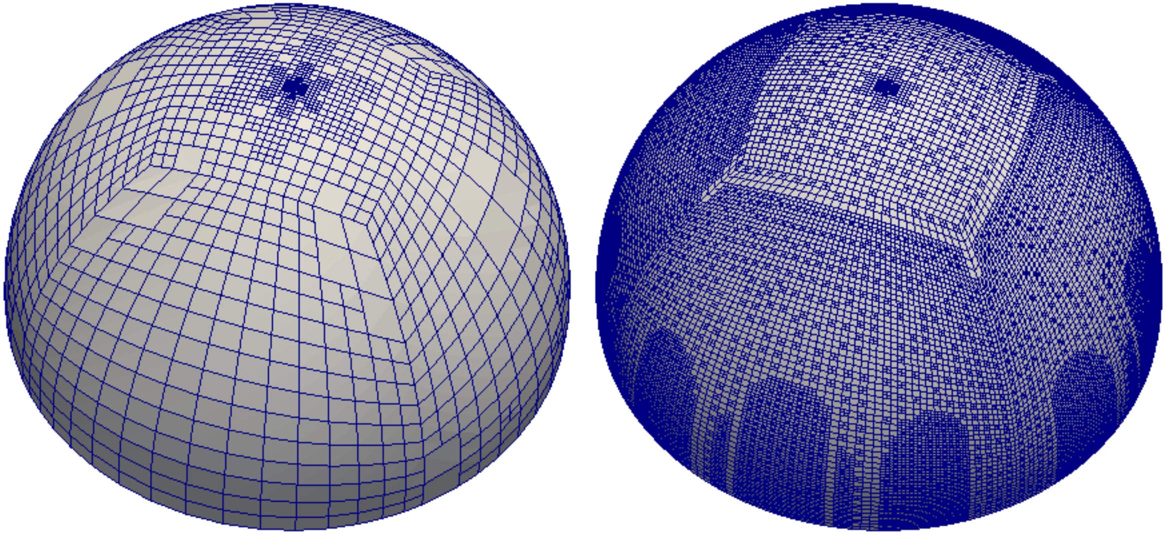
  $$\eta_T \text{ (Galerkin error)}$$

  and

  $$\text{either } \epsilon_T \text{ or } \|\nabla(\mathbf{P} - I_k\mathbf{P})\|_{L_\infty(T)} \text{ (geometric error).}$$

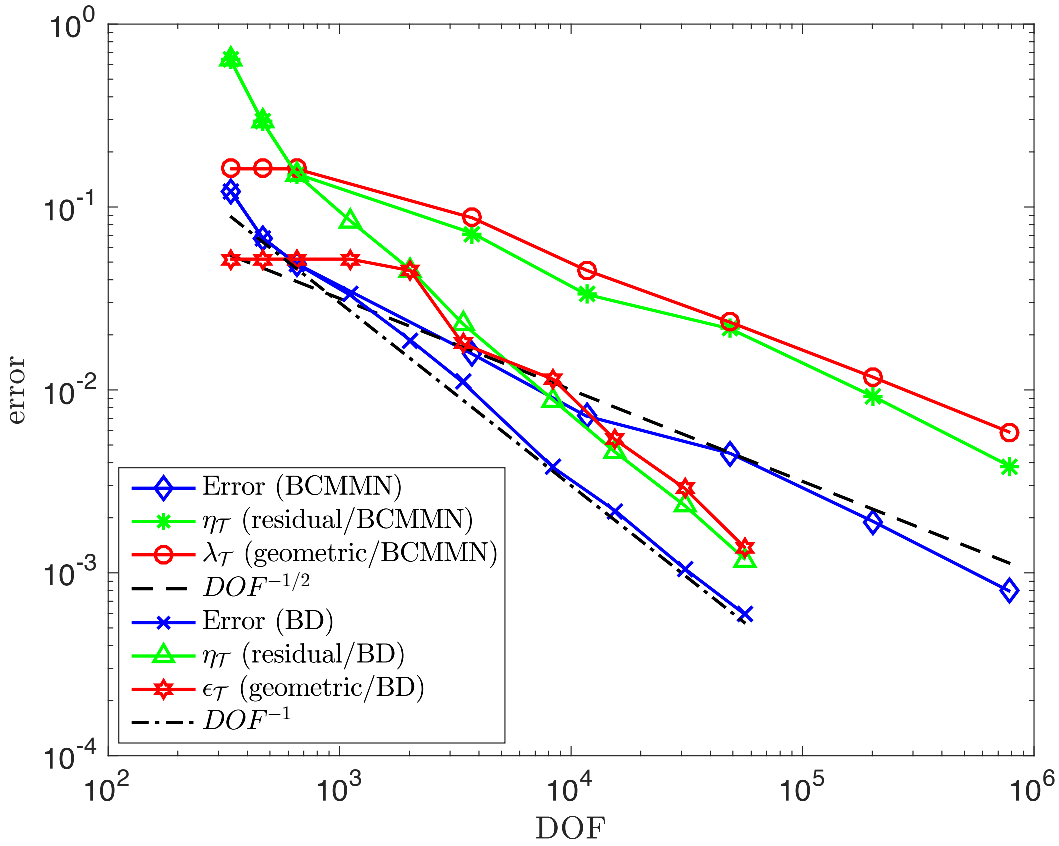
- **Polynomial degree:**. We show results for $r = 2$, $k = 1$.
  (Algorithms perform similarly for isoparametrics ($r = k = 1$)).

Meshes



Adaptive meshes after 10 AFEM iterations with $r = 2$, $k = 1$: BD refinement (left) and BCMMN refinement (right).

Error decrease



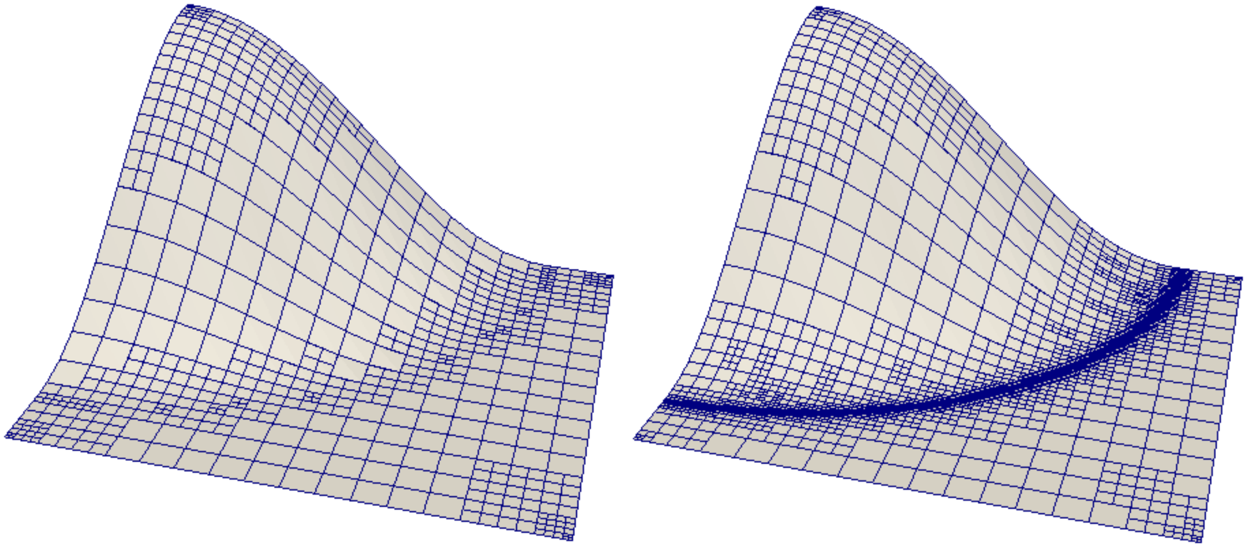
Experiment 2

Setup:

- γ is a $C^{2,\alpha}$ surface ($\alpha = 2/5$).
- $f \equiv 1$, u is unknown.
- Can show: $u \in H^{3-\epsilon}$, any $\epsilon > 0$.
- Three refinement routines:

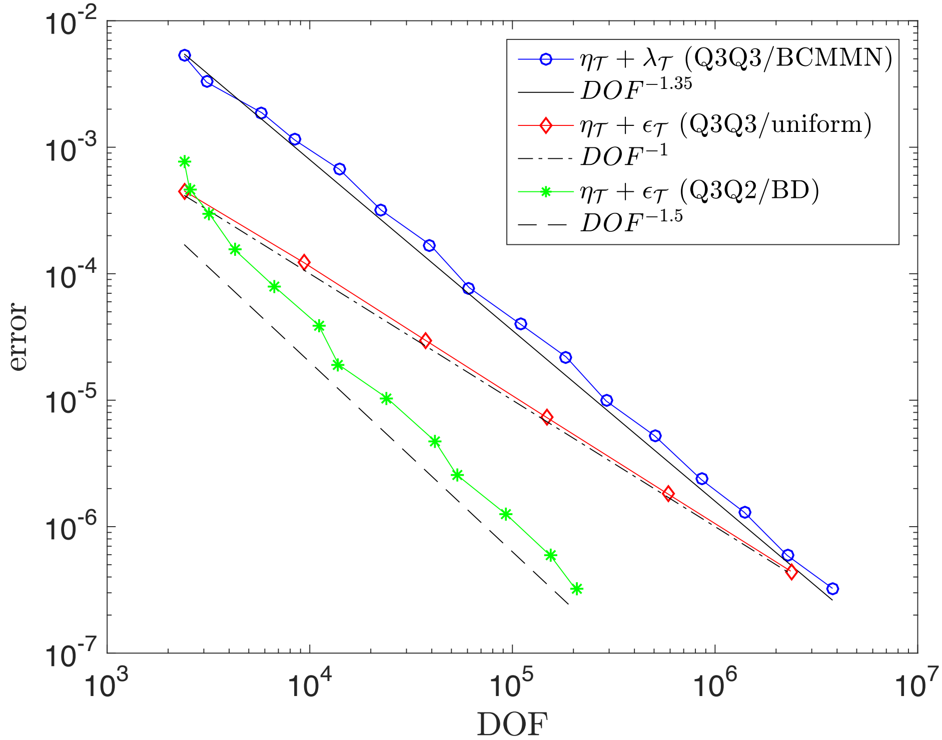
  1. Uniform (measure geometric error w/BD estimator).
  2. AFEM: Q3/Q3 with BCMMN estimator, tolerance 5×10^{-7}.
  3. AFEM: Q3/Q2 with BD estimator, tolerance 5×10^{-7}.

Adaptive meshes



Adaptive meshes after 20 AFEM iterations: BD refinement with $r = 3$, $k = 2$ (left) and BCMMN refinement with $r = k = 3$ (right).

Error decrease



Note: Adaptive error decrease is suboptimal for $Q3Q3$/BCMMN, but NOT for Q3Q2/BD.

Outline

1. Canonical surface FEM
   - Implicit surface representations
   - FEM
   - Consistency errors and a priori estimates
2. Eigenvalue problems
   - Definitions
   - Results of canonical arguments
   - Quadrature-based superconvergence argument
3. A posteriori estimates
   - Estimates assuming canonical implicit representation
   - Parametric representation of surfaces
   - New estimates
4. **Conclusion**

Conclusions

Some future directions:.

1. Nonsmooth (not C^2) surfaces
   - Closest point projection isn't immediately useful.
   - So far, only parametric viewpoint has been used in proofs: $O(h^\alpha)$ geometric error on a $C^{1,\alpha}$ surface.
   - Implies $O(h^\alpha)$ geometric error on $C^{1,\alpha}$ surfaces with $\alpha < 1$, but $O(h^2)$ on C^2 surfaces (!).
2. Vector Laplacians/Stokes on surfaces:
   - Several recent papers: Metatheorem doesn't always hold.
   - For some methods $O(h^{k+1})$ geometric error can be recovered if a better approximation to the normal is used in the definition of the discrete energy inner product.
   - For other methods the situation is less clear.