5 The One-Dimensional Heat Equation

The one-dimensional heat equation describes (among other things) the flow of heat in a body of homogeneous material. We will derive this equation in the next section.

5.1 Derivation of the One-Dimensional Heat Equation

We consider a thin bar of conducting material, which is of length l and such that the lateral sides of the bar are perfectly insulated so that there is no loss of heat. We will assume that the temperature u in the rod depends only on the position x and time t, and not on the lateral co-ordinates y and z, so the temperature along any given cross-section is uniform. This assumption is reasonable based on the choice of the rod which we have chosen to be thin in comparison to the length.

We start the derivation with the *Fourier law* which states that the amount of heat flowing through unit cross-sectional area in the bar per unit time, called the *flux* Q, is given by

$$Q(x,t) = -K\frac{\partial u}{\partial x}(x,t) \tag{1}$$

where K is the heat diffusion constant and depends on the material of the rod, and $u(x,t)$ is the temperature at position x and time t.

Consider an infinitesimal portion of the rod of length $\triangle x$ located between the points x and $x+\triangle x$. The amount of heat flowing into the point x is given by $Q(x,t)$. Similarly, the amount of heat flowing into the other end at $x+\triangle x$ is given by $-Q(x+\triangle x,t)$ (note the sign). The net increase in the differential element (per unit cross-sectional area) in a time $\triangle t$ is given as

$$\text{Increase in heat in element in time } \triangle t = [Q(x,t) - Q(x+\triangle x,t)]\,\triangle t \tag{2}$$

The amount of heat energy per unit cross section in the selected portion of the rod *i.e.,* the differential element of mass $\triangle M$ (and length $\triangle x$) at any time t is given by

$$\sigma\triangle M u \equiv \sigma\rho\triangle x u$$

where σ is the specific heat capacity, ρ is the density of the material and u is the average temperature in the element at time t. In fact we take $u = u(x+\frac{\triangle x}{2},t)$ *i.e.,* u is the temperature at the centre of the element. So we now have

$$\text{Increase in heat in element in time } \triangle t = \sigma\rho\triangle x\triangle t u_t(x+\frac{\triangle x}{2},t) \tag{3}$$

Combining (2) and (3) and dividing both sides by $\triangle x\triangle t$ and taking the limit as $\triangle x \to 0$ we get

$$\lim_{\triangle x\to 0}\frac{Q(x,t)-Q(x+\triangle x,t)}{\triangle x} = \lim_{\triangle x\to 0}\sigma\rho u_t(x+\frac{\triangle x}{2},t),$$

which can be re-written as

$$-Q_x(x,t) = \sigma\rho u_t(x,t),$$

and finally combining with (1) we have

$$Ku_{xx}(x,t) = \sigma\rho u_t(x,t)$$

or

$$u_t = c^2 u_{xx}, \qquad (c^2 = K/\sigma\rho).$$

Note that c^2 is referred to as the *thermal diffusivity.* Note finally that the heat equation is parabolic.

5.2 A Solution of the Heat Equation

The problem we consider is the Heat equation

$$u_t = c^2 u_{xx} \tag{4}$$

with boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0 \tag{5}$$

and initial condition

$$u(x,0) = \phi(x). \tag{6}$$

Using the method of separation of variables *i.e.,* letting

$$u(x,t) = X(x)T(t)$$

we have

$$X'' - kX = 0, \qquad T' - c^2 kT = 0.$$

Note that k has to be chosen such that the boundary conditions are satisfied and the solution obtained for $X(x)$ is non-trivial. It can be shown that k has to be less than zero (please do show why $k \ngeq 0$), thus we let $k = -p^2$ and so

$$X'' + p^2 X = 0, \qquad T' + (cp)^2 T = 0.$$

Solving the ODE in X gives

$$X(x) = A \sin px + B \cos px$$

The boundary conditions for X are $X(0) = 0, X(l) = 0$ which on applying yield

$$X_n(x) = B_n \sin \frac{n\pi x}{l}, \quad n = 1, 2, 3, ...$$

Since $p = \frac{n\pi}{l}$ we thus have for T

$$T' + \lambda_n^2 T = 0, \quad \lambda_n = \frac{cn\pi}{l},$$

which upon solution gives

$$T_n(t) = C_n e^{-\lambda_n^2 t}.$$

Putting the solutions together we have

$$u_n(x,t) = D_n e^{-\lambda_n^2 t} \sin \frac{n\pi x}{l}, \quad n = 1, 2, 3, ... \tag{7}$$

where $\lambda_n = \dfrac{cn\pi}{l}$. Noting that the heat equation is a homogeneous equation, we apply the principle of superposition, which allows us to take a sum of all the solutions corresponding to the values of n thus,

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-\lambda_n^2 t} \sin \frac{n\pi x}{l} \tag{8}$$

Our next aim is to obtain specific solutions that satisfy the initial conditions (6). Essentially, the initial condition determines the unknown coefficient D_n which is a Fourier sine coefficients corresponding to the functions $\phi(x)$ given as

$$D_n = \frac{2}{l} \int_0^l \phi(x) \sin \frac{n\pi x}{l} dx,$$

thus giving a unique solution satisfying the boundary and initial conditions.