

Phase Diffusion Due to Low-Frequency Colored Noise

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Abstract—Phase diffusion in a two-dimensional nonlinear oscillator subject to colored noise is simulated numerically, and compared to theoretical predictions. When the spectrum of the noise sources decays faster than ω^{-2} at high frequencies, standard first-order perturbation results are dominated by second-order effects. A simple analytical model predicts the general form of the diffusion constant in such cases.

Index Terms—Colored noise, diffusion, Monte Carlo simulation, phase noise.

I. INTRODUCTION

NOISE sources in electronic oscillator circuits cause diffusion or jitter of the oscillator phase [1], [2]. This leads to timing jitter in clock circuits and broadening of power spectrum peaks, possibly leading to interference on nearby frequency bands in wireless communications applications. Accurately predicting this effect is therefore an important element of circuit design. In recent work, Demir and co-workers have proposed a general method for predicting phase diffusion in self-sustained oscillators in the presence of white [3] and colored [4] noise sources. In this paper, we show that the effects of colored noise on a simple two-dimensional oscillator model (similar to that proposed by Coram [5]) can be examined by numerical simulation. We compare the phase diffusion in this model to the predictions of Demir's general formula, and we highlight the need for higher order corrections to that formula.

II. OSCILLATOR AND NOISE MODELS

Consider a two-dimensional self-sustained oscillator (with noise sources) described by the equations

$$\begin{aligned}\dot{x} &= f(x, y) + \xi(t) \\ \dot{y} &= g(x, y) + \eta(t)\end{aligned}\quad (1)$$

where f and g are nonlinear functions, and $\xi(t)$ and $\eta(t)$ are independent colored noise sources. The deterministic dynamics is assumed to lead to an asymptotically stable circular limit cycle—a good example is the van der Pol oscillator of frequency Ω given by $f = \Omega y$, $g = -\Omega x + \mu\Omega(1 - x^2)y$, with small nonlinearity parameter: $\mu \ll 1$. The noise sources are independent,

stationary, zero-mean, Gaussian random processes with identical variance κ^2 and correlation function $R(t)$:

$$\langle \xi(t')\xi(t' + t) \rangle = \langle \eta(t')\eta(t' + t) \rangle = \kappa^2 R(t). \quad (2)$$

The noise can also be described through its spectrum $S(\omega)$, which is the Fourier transform of the correlation function

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} R(t) dt. \quad (3)$$

For the van der Pol system with small nonlinearity parameter μ , and other near-circular limit cycle oscillators, the phase dynamics on the limit cycle are easily found using polar coordinates: $x(t) = \rho(t) \cos \theta(t)$, $y(t) = \rho(t) \sin \theta(t)$. Equations (1) then lead to the following equation for the phase angle $\theta(t)$:

$$\frac{d\theta}{dt} = \Omega + \frac{\eta(t)}{\rho} \cos \theta - \frac{\xi(t)}{\rho} \sin \theta \quad (4)$$

where Ω is the (constant) frequency of the oscillator in the absence of noise. We assume for simplicity of this presentation that the limit cycle is strongly attracting, so that fluctuations in ρ are negligible compared to the phase fluctuations—this is a standard assumption for small noise intensities [3], [4]. Moreover, the units of x and y can be chosen so that ρ is set to one. The result is a random differential equation for the limit cycle phase angle $\theta(t)$, with the effects of the colored noises entering in a nonlinear multiplicative fashion

$$\frac{d\theta}{dt} = \Omega + \eta(t) \cos \theta - \xi(t) \sin \theta. \quad (5)$$

The noise terms $\eta(t)$ and $\xi(t)$ are Gaussian, with mean zero and so are fully characterized by their correlation function (2). We use two types of Gaussian noise processes in our simulations, each defined by its correlation function (or equivalent spectrum) as

$$\begin{aligned}R_1(t) &= \exp\left(-\frac{t^2}{2\tau^2}\right) \\ S_1(\omega) &= \frac{\tau}{\sqrt{2\pi}} \exp\left(-\frac{\omega^2 \tau^2}{2}\right) \\ R_2(t) &= \left(1 + \frac{|t|}{\tau}\right) \exp\left(-\frac{|t|}{\tau}\right) \\ S_2(\omega) &= \frac{2\tau}{\pi(1 + \tau^2 \omega^2)^2}.\end{aligned}\quad (6)$$

The correlation function $R_1(t)$ is particularly convenient for numerical simulation, while the second noise process arises naturally as the output of a series of filters with white noise input. In both cases, the parameter τ characterizes the timescale of

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variation of the random process, so the frequency τ^{-1} estimates the bandwidth of the noise. Noting that the parameters Ω and κ both have the dimensions of 1/time, the timescale τ is chosen as a reference and parameters are presented in units of τ^{-1} . We consider only stationary noise processes here, so the spectra must be integrable at low frequencies, unlike, for instance, ideal $1/f$ noise, which cannot be stationary if it has no low-frequency cutoff [6]. Note that both spectrum S_1 and S_2 decay faster than ω^{-2} at high frequencies.

III. NUMERICAL RESULTS

We use Monte Carlo simulations to investigate the diffusion properties of (5), i.e., to find the phase diffusion constant of the oscillator, and hence the phase jitter [4]. Numerical approximations of Gaussian random functions may be constructed using a combination of a large number N of Fourier modes, as follows:

$$\eta(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^N a_n \cos(\omega_n t) + b_n \sin(\omega_n t). \quad (8)$$

The amplitudes a_n and b_n are random numbers from a Gaussian distribution of mean zero and variance κ^2 . The ω_n are random numbers from a distribution shaped as the Fourier transform of $R(t)$ —for the correlation function $R_1(t)$, for instance, this means the ω_n are chosen from a zero-mean Gaussian distribution of variance $1/\tau$. Random functions constructed in this way are Gaussian in the limit $N \rightarrow \infty$ [7]–[9]; we use $N = 100$ in the experiments reported here.

In each realization, the independent noise functions are generated as above. The ordinary differential equation (5) is then solved numerically, with initial condition $\theta(0) = 0$, to find $\theta(t)$. Statistics of $\theta(t)$ are calculated over multiple realizations. The diffusion constant D is defined by the asymptotic relation

$$\text{var}(\theta) \sim Dt \quad \text{as } t \rightarrow \infty \quad (9)$$

where $\text{var}(\theta)$ is the variance of the distribution of phase values at time t

$$\text{var}(\theta) \equiv \langle \theta^2(t) \rangle - \langle \theta(t) \rangle^2. \quad (10)$$

The variance is calculated for times t up to 10τ . A linear fit to the variance for times between 5τ and 10τ gives the diffusion constant D as the slope of the fitted line. Note that to avoid ambiguity in calculating $\text{var}(\theta)$ we consider $\theta(t)$ to have range $(-\infty, \infty)$ rather than the periodic range $[0, 2\pi]$ appropriate to oscillators. For the small noise levels considered here, the phase diffusion effects are independent of the choice of the range of θ .

The phase diffusion constant D is a measure of the noise-induced uncertainty in the phase angle as time increases, and so provides an estimate of the jitter in the phase. Also, phase fluctuations influence the oscillator spectrum, and in particular the variance (9) is important in calculating the spectral lineshape of the oscillator close to the carrier frequency (see (55) and (62) of [4]). It is therefore of considerable interest to be able to accurately predict the value of the diffusion constant D .

The symbols in Fig. 1 denote the calculated phase diffusion constant D for correlation function $R_1(t)$ and noise level $\kappa =$

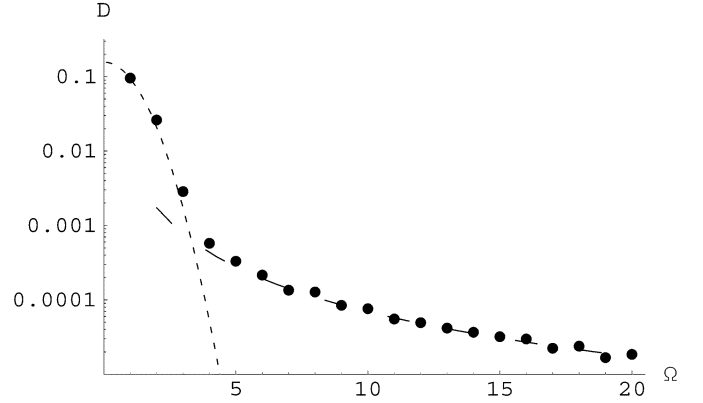


Fig. 1. Dots show the phase diffusion D for noise sources described by (7) as a function of the oscillator frequency Ω , with noise level $\kappa = 0.25\tau^{-1}$, calculated from the phase variance over 10^3 realizations. The short-dashed line shows the approximation (12); the long-dashed line is an empirical fit for the large- Ω form in this case, $D \sim 2.5\tau\kappa^4\Omega^{-2}$. The frequency Ω and the diffusion constant D are each expressed in units of τ^{-1} , note both axes are logarithmic here.

$0.25\tau^{-1}$, and for a variety of values of the oscillator frequency Ω . An approximation, accurate to order κ^2 , to the phase diffusion constant may be calculated according to the method of Demir [4, eq. [(23)]

$$\frac{d}{dt}(\text{var}(\theta)) = 2\kappa^2 \int_0^t \frac{1}{2} [e^{i\Omega(t-s)} + e^{-i\Omega(t-s)}] R(t-s) ds \quad (11)$$

and so in the limit of $t \rightarrow \infty$ we get

$$D = 2\kappa^2 \int_0^\infty \cos(\Omega s) R(s) ds = 2\pi\kappa^2 S(\Omega). \quad (12)$$

For the noise spectrum $S_1(\omega)$ used in the first example, Demir's prediction for the diffusion constant is thus

$$D = \sqrt{2\pi}\kappa^2 \exp\left(-\frac{\Omega^2\tau^2}{2}\right). \quad (13)$$

This is plotted in Fig. 1 with a short-dashed line, note it fits the numerical results well for moderate values of the oscillator frequency Ω . However, at higher values of the frequency Ω , the numerical results indicate that the diffusion constant is much larger than the (exponentially small) value predicted by Demir's formula. Recalling that Ω is plotted in Fig. 1 in units of the noise bandwidth τ^{-1} , this divergence is seen to occur when

$$\Omega \gg \tau^{-1}. \quad (14)$$

At oscillator frequencies beyond the noise bandwidth we find an empirical fit (long-dashed line) which is proportional to $\tau\kappa^4\Omega^{-2}$ —note that at large frequencies a power-law decay of D with Ω is much slower than the exponential decay predicted in (13) and so can dominate, even at small noise intensities κ .

We have shown in this example that colored noise in a simple oscillator causes phase diffusion which matches the formula of Demir [4] at low oscillator frequencies, but seems to deviate at higher frequencies. Our further numerical experiments indicate that this result is generic—when the spectrum $S(\omega)$ of the

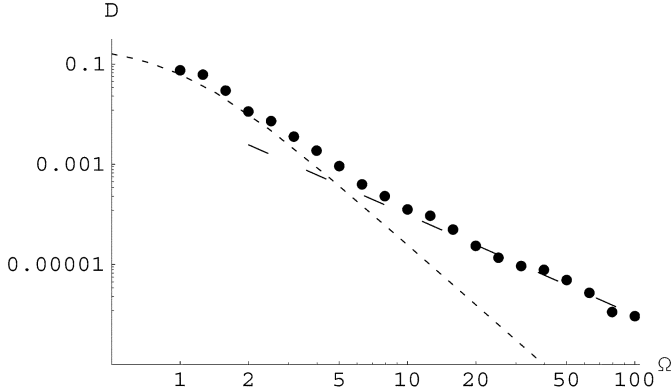


Fig. 2. Dots show the phase diffusion D for noise sources described by (7) as a function of the oscillator frequency Ω , with noise level $\kappa = 0.25 \tau^{-1}$, calculated from the phase variance over 10^3 realizations. The short-dashed line shows the approximation (12); the long-dashed line is an empirical fit for the large- Ω form in this case, $D \sim 2.5\tau\kappa^4/\Omega^2$. The frequency Ω and the diffusion constant D are each expressed in units of τ^{-1} , note both axes are logarithmic here.

colored noise processes $\xi(t)$ and $\eta(t)$ decays faster than ω^{-2} , a power-law scaling of the phase diffusion with Ω emerges at high oscillator frequencies ($\Omega \gg 1/\tau$). Fig. 2 shows the dependence of the phase diffusion on the oscillator frequency Ω when the noise has correlation function $R_2(t)$ [defined in (7)]. Again, Demir's approximation (12) works well at low frequencies, but a power-law fit $D \sim 2.5\tau\kappa^4/\Omega^2$ better describes the phase diffusion at high oscillation frequencies. This effect is not captured by first-order (order κ^2) perturbation methods such as Demir's, but the Ω^{-2} scaling may be shown to arise as a general second-order (i.e., order κ^4) effect using higher order perturbation calculations [10]. The high-frequency results are therefore not artifacts of the numerical simulation: in the next section we also use a simple analytical model to derive the main scaling properties of D when the oscillator frequency is much higher than the noise bandwidth. We note that colored noise processes of the type considered here (faster than ω^{-2} spectral decay) can arise in many ways, for instance from filtered Ornstein-Uhlenbeck [11] or white noise processes. It seems likely that this second-order effect will also be important in the high-frequency dynamics of more complex oscillators than the simple model (5) presented here.

IV. ANALYTICAL MODEL

An analytical result giving the κ^4/Ω^2 scaling of the phase diffusion for an approximate model is examined here. We consider high frequency oscillations, in particular assuming $\Omega \gg \tau^{-1}$, so that many oscillator cycles are completed within the characteristic timescale τ of the noise. In this case, we replace the continuous noise processes $\eta(t)$ and $\xi(t)$ with random processes which are constant over time intervals T (with $T \gg \Omega^{-1}$), changing their value only at discrete times $t = nT$, where n is an integer. The values taken by the processes in each T -interval are chosen from a Gaussian distribution of mean zero and variance κ^2 . Within each T -interval the values of ξ and η are then frozen,

and it is possible to find the period T_θ of the limit cycle oscillation from (5) as

$$\int_0^{T_\theta} dt = \int_\psi^{\psi+2\pi} \frac{1}{\Omega + \eta \cos \theta - \xi \sin \theta} d\theta$$

$$\Rightarrow T_\theta = \frac{2\pi}{\sqrt{\Omega^2 - \kappa^2 r^2}}. \quad (15)$$

The dimensionless random variable r is defined as $r = \sqrt{\xi^2 + \eta^2}/\kappa$; note that the probability distribution function (PDF) of r can be obtained from the Gaussian PDF's of the noise sources ξ and η as $P(r) = r \exp(-r^2/2)$.

In each interval T of constant noise values, the average phase speed (over many oscillator cycles) is therefore $\nu = 2\pi/T_\theta = \sqrt{\Omega^2 - \kappa^2 r^2}$. Note the small-noise assumption here that $\kappa \ll \Omega$ means the square root for ν is well defined for all except a negligible fraction of the values of r ; the latter correspond to cases where the oscillator becomes phase-locked ($\nu = 0$) for the duration of the T -interval. The angular position $\theta(t)$ at time $t = nT$ is given in this model by the sum of the displacements over each T -interval

$$\theta(t) = T \sum_{j=1}^n \nu_j \quad (16)$$

where $\nu_j = \sqrt{\Omega^2 - \kappa^2 r_j^2}$ is the average speed during the j th T -interval. As the ν_j values in different intervals are independent, it is easy to show that the variance of $\theta(t)$ is related to the variance of the ν_j by

$$\text{var}(\theta) = nT^2 \text{var}(\nu) \quad (17)$$

and so the diffusion constant is $D = \text{var}(\theta)/t = T \text{var}(\nu)$. The variance of ν is calculated using the PDF of r as

$$\text{var}(\nu) = \langle \nu^2 \rangle - \langle \nu \rangle^2$$

$$= \int_0^{\frac{\Omega}{\kappa}} P(r) (\Omega^2 - \kappa^2 r^2) dr$$

$$- \left[\int_0^{\frac{\Omega}{\kappa}} P(r) \sqrt{\Omega^2 - \kappa^2 r^2} dr \right]^2. \quad (18)$$

These integrals may be expressed in closed form, however the main point of interest is the behavior of D in the physically relevant limit of small noise intensity and high oscillator frequency. An expansion in terms of the small parameter κ/Ω leads to the result

$$D \sim T \frac{\kappa^4}{\Omega^2} + O\left(\frac{\kappa^6}{\Omega^4}\right). \quad (19)$$

The scaling of the diffusion constant with frequency as Ω^{-2} , and with noise intensity as κ^4 in this simplified model is consistent with that observed in the numerical results for the continuous-noise case. A focus of our current research [10] is to relate an effective timescale T in the continuous-noise case to the form of the correlation function $R(t)$ of the noise processes.

TABLE I
COMPARISON OF EFFECTIVE TIMESCALES AND NOISE CORRELATION TIMES
FOR TWO NOISE PROCESSES

$R(t)$	T_{eff} (fitted)	$\tau_c = \int_0^\infty R(t) dt$	T_{eff}/τ_c
$R_1(t)$	1.8τ	1.25τ	1.43
$R_2(t)$	2.5τ	2τ	1.25

V. DISCUSSION

The numerical results of Section III show that when the noise spectrum $S(\omega)$ decays faster than ω^{-2} , the phase diffusion constant D scales as

$$D \sim T_{\text{eff}} \kappa^4 \Omega^{-2} \quad (20)$$

at high oscillator frequency Ω , with T_{eff} having the dimensions of time. The analysis of Section IV confirms that such scaling occurs if the continuous noise processes are replaced by piecewise-constant random functions with discontinuous changes at regular time intervals T_{eff} . The relationship between the correlation function $R(t)$ of the continuous noise and the “effective” time interval T_{eff} defined by fitting the numerical results for D to (20) remains to be clarified. This relationship is nontrivial; for instance it is shown in Table I that T_{eff} is not simply proportional to the correlation time of the noise (defined in the usual way as $\tau_c = \int_0^\infty R(t) dt$): note that the ratio in the final column of the table does not have the same value for the two example noise processes (6) and (7). Further work on second-order perturbation theory is underway [10] to enable the prediction of T_{eff} directly from the noise correlation function $R(t)$, thus providing an accurate value for the phase diffusion at high oscillator frequencies. Also of interest for further work is the effect on the present results of relaxing the assumption of strong amplitude control which leads from (4) to (5). Although the results presented here are focussed only on the simple phase oscillator (5), similar effects are expected in any oscillator where the noise influence terms [i.e., the $\cos \theta$ and $-\sin \theta$ terms multiplying the noise sources in (5)] average to zero over the limit cycle.

Previous simulations by Kaertner [1] and Demir [4] have focused exclusively on noise spectra which fall off as ω^{-2} or ω^{-1} . For such processes, the second-order effect we highlight here remains subdominant to the first-order result (12). We confirm this (and check the validity of the Monte Carlo simulation scheme) for a ω^{-2} decay in Fig. 3, where the diffusion constant for noise with spectrum $S_3(\omega) = \tau/\pi(1 + \omega^2\tau^2)$ is seen to closely match Demir’s formula (12). Thus, the results presented here do not contradict previous simulations, but highlight the importance of

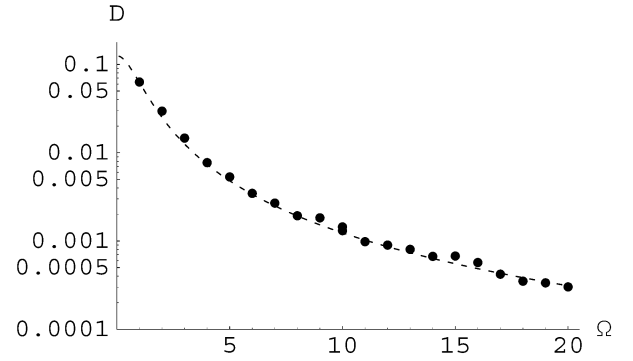


Fig. 3. Dots show the phase diffusion D for noise sources with spectrum $S_3(\omega) = \tau/\pi(1 + \omega^2\tau^2)$ as a function of the oscillator frequency Ω , with noise level $\kappa = 0.25 \tau^{-1}$, calculated from the phase variance over 10^3 realizations. The short-dashed line shows Demir’s first-order approximation (12). The frequency Ω and the diffusion constant D are each expressed in units of τ^{-1} .

second-order effects when the noise spectrum decays faster than ω^{-2} at high frequencies.

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